THE K-THEORY SPECTRUM OF VARIETIES

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ABSTRACT. Using a construction closely related to Waldhausen's $S_\bullet$-construction, we produce a spectrum $K(\mathbf{Var}_{/k})$ whose components model the Grothendieck ring of varieties (over a field k) $K_0(\mathbf{Var}_{/k})$. We then produce liftings of various motivic measures to spectrum-level maps, including maps into Waldhausen's K-theory of spaces $A(*)$ and to $K(\mathbf{Q})$. We end with a conjecture relating $K(\mathbf{Var}_{/k})$ and the doubly-iterated K-theory of the sphere spectrum.

1. INTRODUCTION

1.1. **Background.** Let k be a field. The Grothendieck ring of varieties $K_0(\mathbf{Var}_{/k})$ is defined first as a group with generators isomorphism classes $[X]$ where X is a variety and relations $[X - Y] + [Y] = [X]$ where $Y \hookrightarrow X$ is a closed inclusion. The multiplication is induced by Cartesian product of varieties. This ring is a fundamental object of study for algebraic geometers: it is a universal home for Euler characteristics of varieties, called motivic measures, as well as an easy version of motives.

There are many reasons for interest in the ring $K_0(\mathbf{Var}_{/k})$. From a classical perspective, one reason to consider $K_0(\mathbf{Var}_{/k})$ is that for k a finite field, this is a home for the point-counting map. That is, if we let $\mathbf{pc}(X)$ denote the set of k-points of a k-variety X, then we obtain a ring homomorphism $K_0(\mathbf{Var}_{/k}) \to \mathbf{Z}$ via $X \mapsto \#\mathbf{pc}(X)$.

The Grothendieck ring of varieties also arise as the target for "motivic integration" [15]. Motivic integration was invented by Kontsevich as a method for producing rational invariants of Calabi-Yau varieties. In his setup, the target for such an integral is a ring closely related to $K_0(\mathbf{Var}_{/k})$. In general, any ring homomorphism $K_0(\mathbf{Var}_{/k}) \to A$ can be used as a measure for motivic integration, hence the term *motivic measure*.

A number of authors have constructed interesting motivic measures. For example, in [14] the authors construct a motivic measure $K_0(\mathbf{Var}_{/k}) \to \mathbf{Z}[SB]$ where the latter denotes the free group ring on stable birational classes of varieties. Furthermore, they show that the kernel of that ring map is the ideal generated by the class of the affine line $[\mathbf{A}^1_k]$. This, of course, indicates a deep relationship between $K_0(\mathbf{Var}_{/k})$ and stable birational geometry. The motivic measure was used in [14] to prove the irrationality of a certain motivic zeta function. Another slightly more exotic motivic measure was produced in [9]: a ring map $K_0(\mathbf{Var}_{/k}) \to K_0(\mathbf{PT})$ where $\mathbf{PT}$ is the category of small pre-triangulated categories.

As with other Grothendieck rings, one should expect that this ring arises as the connected components of some space or spectrum. Indeed, Zakharevich [24] showed, using her formalism of assemblers, that this is the case and used the result to prove a number of results relating to cut-paste conjectures. We will call the spectrum she defined $K(\mathbf{Var}_{/k})$.

We are interested in finding maps out of $K(\mathbf{Var}_{/k})$, which can be thought of as "derived" motivic measures. Not only would this allow a generalization of motivic integration, but it would allow us to better understand the structure of the spectrum $K(\mathbf{Var}_{/k})$. The approach of constructing $K(\mathbf{Var}_{/k})$ via assemblers makes producing maps out of it somewhat difficult. While assemblers work for a quite general class of cutting and pasting problems, for our more narrow interests, constructions related to the classical $S_\bullet$ and Q-constructions will be easier to use. Much of the work in this paper is setting up the structure necessary to define an appropriate Waldhausen-style construction of $K(\mathbf{Var}_{/k})$.

Once this is accomplished, we obtain a number of maps out of $K(\mathbf{Var}_{/k})$ which will allow us to probe the deeper structure of the category of varieties.

1.2. **Results.** The most general method for defining K-theory is via Waldhausen categories [22]. These are categories equipped with maps called cofibrations and weak equivalences satisfying certain axioms. In seeking to define a putative $K(\mathbf{Var}_{/k})$ in this context, one immediately runs into the problems — for example, $\mathbf{Var}_{/k}$ cannot be a Waldhausen category since it has no zero object, nor does it have quotients or pushouts in general. However, these objections can be remedied.

First, one may show that $\mathbf{Var}_{/k}$ has just enough pushouts: pushouts where both legs are closed inclusions exist. Also, "quotients" in our setting will be "subtractoin," $Y - X$ for closed inclusions $X \hookrightarrow Y$ of varieties. One may also observe that a zero object is not actually needed, but an initial object is and the empty variety will work in this case. In Section 2 we verify the various technical details that will make Waldhausen's construction work for $\mathbf{Var}_{/k}$, and in Section 3 we then follow Waldhausen and obtain the following result, originally due to Zakharevich.

Theorem 1.1. [24] *One can define a spectrum $K(\mathbf{Var}_{/k})$ such that*

$$\pi_0 K(\mathbf{Var}_{/k}) = K_0(\mathbf{Var}_{/k})$$

Once we have $K(\mathbf{Var}_{/k})$ defined as a kind of $S_\bullet$-construction, one may use standard techniques to recognize structure of that spectrum. For example, $\mathbf{Var}_{/k}$ is obviously a symmetric monoidal category under cartesian products, so one could expect that $K(\mathbf{Var}_{/k})$ inherits some of that structure. This is indeed true.

Theorem 1.2. *$K(\mathbf{Var}_{/k})$ is an E_∞-ring spectrum.*

Under rather specific assumptions about the underlying categories, there are various equivalences and cofiber sequences that K-theory satisfies: the approximation theorem [22], localization [18, 22], and Devissage [18]. The category of varieties is remarkably poorly behaved, however, and it is unlikely that the most powerful of the K-theory theorems could be proved for it. It is nevertheless true that a specific case of devissage holds: any variety can be filtered by smooth varieties and the corresponding devissage statement on K-theory obtains. In order to prove this we first have to introduce a version of Quillen's Q-construction that will allow us to mimic Quillen's proof of devissage for exact categories. Once we have this, one may state the following.

Theorem 1.3 ("Devissage")**.** *There is a weak equivalence of spaces*

$$K(\mathbf{Var}^{sm}_{/k}) \simeq K(\mathbf{Var}_{/k})$$

induced by the inclusion of categories $\mathbf{Var}^{sm}_{/k} \hookrightarrow \mathbf{Var}_{/k}$.

Remark 1.1. The above theorem is a special case of Zakharevich's [24]. However, the proof is quite different. It is necessary to have a proof in this model in order to produce maps out of $K(\mathbf{Var}^{\mathrm{sm}}_{/k})$.

Now that the machinery is set up, one may begin to define maps in and out of $K(\mathbf{Var}_{/k})$. These may be considered to be "derived" versions of motivic measures.

First, one can define a model for the unit map $S \to K(\mathbf{Var}_{/k})$. Next, when k is finite, a point-counting functor defines a map from $K(\mathbf{Var}_{/k})$ to the sphere spectrum. One may also consider a complex variety as a topological space and relate this to Waldhausen's K-theory of spaces, $A(*)$ [22]. Finally, one can define maps to $K(\mathbf{Q})$ by using derived versions of the Euler characteristics. Summarizing, we have

Theorem 1.4. *There are maps (and explicit models for them)*

(1) $S \to K(\mathbf{Var}_{/k})$.
(2) $K(\mathbf{Var}_{/k}) \to S$, *$k$ finite.*
(3) $K(\mathbf{Var}_{/\mathbf{C}}) \to A(*)$.
(4) $K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathrm{Ch}^b(\mathbf{Q}))$ *and* $K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathbf{Q})$.

In fact, there should be maps from $K(\mathbf{Var}_{/k})$ into much "larger" ring spectra. Instead of discarding information by simply counting points or taking cohomology, one could instead pass to derived categories. This would give us a conjectural map $K(\mathbf{Var}_{/k}) \to K(\mathcal{C}\mathrm{at}^{\mathrm{Ex}})$ where $\mathcal{C}\mathrm{at}^{Ex}_{\infty}$ is the ∞-category of stable ∞-categories [3]. A more concrete manifestion of this map is the following conjecture.

Conjecture 1. *There is a map* $K(\mathbf{Var}_{/k}) \to K(K(S))$ *or* $K(K(k))$ *of* E_∞*-ring spectra.*

As we will see later, this would be true if the following holds.

Conjecture 2. *Let X be a variety over a regular ring. Then* $K(X) := K(\mathbf{QCoh}(X))$ *is compact as a* $K(S)$*-module or* $K(X)$ *is compact as a* $K(k)$*-module.*

Remark 1.2. The conjectures above would essentially supply a lift of of the Bondal-Larsen-Lunts motivic measure $K_0(\mathbf{Var}_{/k}) \to K_0(\mathbf{PT})$.

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2. Scheme Theoretic Preliminaries

In topological contexts, the construction of K-theory via Waldhausen categories [22], depends heavily on having certain categorical limits and colimits. We cannot take for granted the existences of all (or any) limits and colimits in the category of varieties. However, in this section we show that all of the limits and colimits that will be necessary do, in fact, exist. We rely on Schwede's paper on pushouts in schemes [19] for this.

The results which we will need from Schwede are the following two theorems.

Theorem 2.1 (Schwede). *Suppose A, B are rings and suppose $I \subset A$ is an ideal, and that there exists a map $f : B \to A/I$. Consider the diagram*

$$\begin{array}{ccc} Z = \operatorname{Spec} A/I & \longrightarrow & X = \operatorname{Spec} B \\ \downarrow & & \\ Y = \operatorname{Spec} A & & \end{array}$$

Then,

1. *The pushout $X \amalg_Z Y$ exists and is affine*
2. *$Y \to X \amalg_Z Y$ is a closed immersion*
3. *both $X \to X \amalg_Z Y$ and $Y \to X \amalg_Z Y$ are morphisms of schemes.*

Theorem 2.2 (Schwede). *Let $Z \hookrightarrow X$ and $Z \hookrightarrow Y$ be closed immersions of schemes. Then $X \amalg_Z Y$ exists.*

Remark 2.1. It will be useful for us to record how the pushout is constructed locally. In the proof of the first theorem, it turns out that $X \amalg_Z Y = \operatorname{Spec} C$ where

$$C = B \times_{A/I} A = \{(a, b) : f(b) = a + I\}$$

Although not stated explicitly in Schwede, the following is a consequence of the proof of Th.2.2.

Corollary 2.1. *In the situation of Th.2.2, $X \to X \amalg_Z Y$ and $Y \to X \amalg_Z Y$ are closed immersions.*

Remark 2.2. This is the statement closed inclusions are preserved by cobase change.

However, we will need more. Since we will be working just in the category of varieties, we need that in fact pushouts exists in that category.

Proposition 2.1. *Let $Z \to X$, $Z \to Y$ be closed embeddings of varieties. Then $X \amalg_Z Y$ is a variety.*

Proof. We need to show that $X \amalg_Z Y$ is reduced, separated, and finitely generated. The reducedness is a completely local question, and so follows from the fact that X, Y, Z are all varieties. To see that it is separated, apply the valuative criterion for separatedness. The fact that $X \amalg_Z Y$ is finitely generated is local, and thus the content of the following easy lemma. □

Lemma 2.1. *Let $g : C \to A$ and $f : B \to A$ be surjective morphisms of commutative rings with A, B, C Noetherian. Then the fiber product $A \times_B C$ is also Noetherian. In particular, if A, B, C are finitely generated k-algebras, then $B \times_A C$ is a finitely generated k-algebra.*

Proof. Let $\pi_B : B \times_A C \to B$ and $\pi_C : B \times_A C \to C$ be the obvious projections. By inspection, these are surjective. Now, $\ker(\pi_C) = \ker(B \to A)$ and $\ker(\pi_B) = \ker(C \to A)$. Furthermore, $\ker(\pi_B) \cap \ker(\pi_C) = 0$ in $B \times_A C$. Also, $(B \times_A C)/\ker \pi_B \cong A$ and $(B \times_A C)/\ker \pi_C \cong B$ which are both Noetherian. These two facts together show that $B \times_A C$ is Noetherian. □

Remark 2.3. The paper [19] shows how pushouts can fail spectactularly to exist if both legs of the pushout are not closed inclusions.

We now have to examine the interaction of categorical pushouts with the operation of "subtraction," which as stated in the introduction will be our analogue of quotients. In typical Waldhausen categories, quotients and colimits automatically commute since colimits commute. In the categories we will deal with in the paper, this will not happen: we have to prove by hand that subtraction and various colimits commute.

In the category of varieties, an "exact sequence" will be a sequence

$$X \hookrightarrow Y \hookleftarrow Y - X$$

where the first map is a closed embedding. Intuitively, we would like to think of this as a pushout along a map $X \to \emptyset$. Of course, $\emptyset$ is not a final object in $\mathbf{Var}_{/k}$, so we can't take the interpretation literally. However, will will show that such a subtraction operation commutes with all of the colimits that we will need, and thus we will be able to do our usual constructions.

We begin by precisely defining subtraction in the category of schemes (and varieties).

Definition 2.1. Let $i : Z \hookrightarrow X$ be a closed immersion. We define $X -_i Z$ as follows. The immersion i determines a homeomorphism onto a closed subset $i(Z) \subset X$, which in turn determines an open subset $X - i(Z)$ of X. To view this as a scheme, we restrict the structure sheaf $\mathcal{O}_X$ to $X - i(Z)$. That is

$$X -_i Z = (X - i(Z), \mathcal{O}_X|_{X-i(Z)}).$$

Remark 2.4. We need a number of remarks about the degree of uniqueness we will be requiring from subtractions.

The above says that given a particular closed immersion $i : Z \hookrightarrow X$, there is a *canonically determined* object $X -_i Z$. There are, however, other closed immersions, since we can pick any isomorphism $Z' \cong Z$ and $i' : Z' \hookrightarrow X$ will also be a closed immersion. In this case $X -_{i'} Z'$ and $X -_i Z$ will be *equal*. One can view this as another version of the fact that if Z, Z' are isomorhic as schemes, then their respective ideal sheaves $\mathcal{I}$ and $\mathcal{I}'$ in $\mathcal{O}_X$ are in fact *equal*.

This all said, there are of course other objects isomorphic to $X -_i Z$. This presents us with a (mild) problem as to how to define "exact sequences." It is too rigid for us to define exact sequences only to be sequences $X \to Z \leftarrow X -_i Z$. This motivates the following definition.

Definition 2.2. Given a closed immersion $i : Z \hookrightarrow X$ define $X - Z$ to be any scheme isomorphic to $X -_i Z$. There will thus be a unique open immersion $X - Z \to X$ that factors through the open inclusion $X -_i Z \hookrightarrow X$.

This allows us the following definition.

Definition 2.3. A pair of maps

$$Z \xrightarrow{\ i\ } X \xleftarrow{\ j\ } Y$$

is an **exact sequence** if i is a closed immersion, j is an open immersion and $Y \cong X -_i Z$.

Now that we have precisely defined subtraction, we move on to showing that the operations behaves well when mixed with various colimits.

Proposition 2.2. *Suppose $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are closed immersions. Then*

$$Y - X \to Z - X$$

is a closed immersion.

Proof. Whether or not this is a closed immersion is a local question. Choose affine opens so that the sequence of inclusions becomes

$$\operatorname{Spec} A/J \to \operatorname{Spec} A/I \to \operatorname{Spec} A.$$

where $J \supset I$.

At this level, $Y - X$ is $\operatorname{Spec}(A/I)_J$ and $Z - X$ is $\operatorname{Spec} A_J$. Since localization commutes with quotients, it's clear that

$$\operatorname{Spec}(A/I)_J \to \operatorname{Spec} A_J$$

is a closed immersion. □

We need a slightly more general version of this proposition. However, we make note right now of a severe issue that arises. In schemes, there is a problem with mapping out of what we're thinking of as "quotients." That is, if we have $X \hookrightarrow Y$ with a quotient $Y - X$ and $X' \hookrightarrow Y'$ with a quotient $Y' - X'$, if there are maps $X \to X'$ and $Y \to Y'$ there is no reason why $X - Y \to X' - Y'$ should even be a map. Thus, the extra condition imposed in the following proposition.

Proposition 2.3. *Suppose we have a diagram of closed immersions of schemes/varieties*

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & Z \end{array}$$

such that W is the scheme-theoretic intersection of Y and X in Z. Then the unique map

$$Y - W \to Z - X$$

is a closed immersion.

Proof. Again, localize until we are dealing with a diagram

$$\begin{array}{ccc} \operatorname{Spec} A/(I+J) & \longrightarrow & \operatorname{Spec} A/J \\ \downarrow & & \downarrow \\ \operatorname{Spec} A/I & \longrightarrow & \operatorname{Spec} A \end{array}$$

Then $Y - W$ is represented as $\operatorname{Spec}(A/I)_{I+J}$ and $Z - X$ as $\operatorname{Spec} A_J$. $(A/I)_{I+J} \cong A_J/I^e$, so the map is a closed immersion. □

Finally, we need to show that "quotients" and the few pushouts that exist commute.

Proposition 2.4. *Given a diagram of schemes*

$$\begin{array}{ccc} D & \hookleftarrow & C \\ \uparrow & & \uparrow \\ B & \hookleftarrow & A \\ \downarrow & & \downarrow \\ B' & \hookleftarrow & A' \end{array}$$

where A is the scheme theoretic intersection of C and B in D, and also the scheme theoretic intersection of B and A' in B', then

$$(B' - A') \amalg_{B-A} (D - C) \cong B' \amalg_B D - A' \amalg_A D$$

canonically.

Proof. Note that the maps $B - A \to B' - A'$ and $B - A \to D - C$ are closed inclusions by the previous proposition, so the pushouts in question make sense.

It suffices to consider the problem affine locally. Choose $B = \operatorname{Spec} R$ — then the following diagram is forced upon us:

$$\begin{array}{ccccc} R_L & \longleftarrow & R/J & \longrightarrow & R/L \\ \downarrow & & \downarrow & & \downarrow \\ R_L/I^e = R_K/I^e & \longleftarrow & R/I & \longrightarrow & R/I \otimes_R R/K = R/I \otimes_{R/J} R/L \\ \uparrow & & \uparrow & & \uparrow \\ R_K & \longleftarrow & R & \longrightarrow & R/K \end{array}$$

where $J \subset I$, $J \subset L$ and the left hand column is made of up localizations. Note that $R/(I + K) = R/(I + L)$ by assumption. We now have a quotient map

$$R \times_{R/I} R/J \to R/K \times_{R/(I+K)} R/L = R/K \times_{R/(I+L)} R/L.$$

The kernel of this projection is the image of the ideal $K \times L$ in $R \times_{R/I} R/J$, which is $K \times_{R/I} L$. Localizing with respect to this ideal we obtain $R_L \times_{R_K/I^e} R_K$, which is what we wanted to show.

□

Finally, we need a trivial fact that will be useful for arguments below.

Lemma 2.2. *Given a pushout diagram of schemes*

$$\begin{array}{ccc} W & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X \amalg_W Y \end{array}$$

there is an isomorphism

$$X - W \xrightarrow{\cong} X \amalg_W Y - Y$$

Proof. This follows from [19, Thm. 3.3] and gluing. □

This is all we will need to know about the interaction of pushouts and varieties.

3. The K-Theory Spectrum of Varieties

In this section we show that we can produce K-theory from a structure slighty different from a Waldhausen category. The category $\mathbf{Var}_{/k}$ is simply not a Waldhausen category, but if one modifies the definition, a suitable replacement notion can be found and a delooping produced. The path to all of this is via "additivity."

The standard slogan for algebraic K-theory is that it is a "universal additive invariant" [3, 2]. That is, it is initial among invariants of categories that split exact sequences. In [22], Waldhausen shows his definition satisfies additivity and then uses that property to produce a delooping of the algebraic K-theory space. Of course, Waldhausen's constructions and proofs use only Waldhausen categories, which have certain category theoretic requirements that $\mathbf{Var}_{/k}$ does not satisfy. We introduce a modification of Waldhausen categories that will allow us to to define an $S_\bullet$-construction for $\mathbf{Var}_{/k}$, and thus $K(\mathbf{Var}_{/k})$. After producing this $S_\bullet$-construction, we present a version of it that allows for immediate recognition of $K(\mathbf{Var}_{/k})$ as a symmetric spectrum, and then a version that allows for recognition of multiplicative properties.

After defining the appropriate $S_\bullet$ constructions, we go on to prove additivity in this context, following McCarthy's [17] method. From there, modifications of a few more theorems of Waldhausen are needed to show that we can deloop the K-theory of varieties to produce an infinite loop space, and thus produce a quasi-fibrant symmetric spectrum.

3.1. **Modified Categories with Cofibrations.** To begin we define the version of categories with cofibration [22] we will work with.

Definition 3.1. A **modified category with cofibrations** is a category $\mathcal{C}$, equipped with a subcategory of cofibrations, $\mathbf{co}(\mathcal{C})$, which satisfies six axioms

(1) There is an initial object, typically referred to as the empty object, $\emptyset$.
(2) Isomorphisms are cofibrations
(3) (**pushouts**)Pushouts along diagrams where both legs are cofibrations exist and cobase change holds. Furthermore, pushout diagrams of this form are also pullback.
(4) (**pullbacks**) Pullbacks along diagrams where both legs are cofibrations exist and satisfy base change. Furthermore, given a pullback diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

where $X \to Z$ and $Y \to Z$ are cofibrations, the pushout-product map $X \amalg_W Y \to Z$ is a cofibration.
(5) (**subtraction**)
   (a) There is a notion of subtraction: Given a cofibration $i : X \hookrightarrow Y$, there is an object Z such that $Z \hookrightarrow Y$ and $Z \cup X = Y$. More precisley, there is a forgetful functor $u : \mathcal{C} \to \mathcal{S}\mathrm{et}$ and for a cofibration $X \hookrightarrow Y$ there is a Z such that $u(Z) = u(Y) - u(i(X))$. The object Z is unique up to unique isomorphism. That is, if Z' is another object in $\mathcal{C}$ satisfying the defining property of subtraction, then there is a unique map $Z' \to Z$ that factors $Z \to Y$.

(b) Subtraction is no functorial on the category of pairs, but it should satisfy the condition that given a pullback square

$$\begin{array}{ccc} X \times_{Y'} X' & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \hookrightarrow & Y' \end{array}$$

there is a unique map

$$Y - (X \times_{Y'} X') \to Y' - X'$$

which is a cofibration and which makes the diagram

$$\begin{array}{ccc} Y & \longleftarrow & Y - (X \times_{Y'} X') \\ \downarrow & & \downarrow \\ Y' & \longleftarrow & Y' - X' \end{array}$$

(6) (**subtraction and pushouts**) Given a diagram

$$\begin{array}{ccccc} X & \hookleftarrow & W & \hookrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \hookleftarrow & W' & \hookrightarrow & Y' \end{array}$$

such that all maps are cofibrations and $W \cong X \times_{X'} W'$ and $W \cong Y \times_{Y'} W'$ we have that

$$(X' - X) \amalg_{(W'-W)} (Y' - Y) \cong (X \amalg_W Y) - (X' \amalg_{W'} Y')$$

Remark 3.1. It will be seen later that condition (2) above is exactly what is needed to make McCarthy's proof of additivity work.

Remark 3.2. Some explanation is needed on the functorality of subtraction. One would like to define a subtraction functor on a category of pairs (X, Y) where $X \hookrightarrow Y$ is a closed inclusion. Then issue then arises of what a map $(X, Y) \to (X', Y')$ should be. First, it seems reasonable that $X \to X'$ and $Y \to Y'$ should be cofibrations. This essentially forces X to be in the intersection (in Y') of X and Y. However, in order for $Y - X \to Y' - X'$ to even make sense, it must be the *whole* intersection. This is why we only define the induced maps between pairs of spaces to exist when X is the pullback.

We also need an appropriate notion of functor between two modified categories with cofibrations.

Definition 3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of modified categories with cofibration. We call such a functor **exact** if F satisfies the following

(1) F preserves the empty variety: $F(\emptyset) = \emptyset$.
(2) F preserves cofibrations
(3) F preserves cofiber sequences, that is, if $X \hookrightarrow Y$ is a cofibration, then $F(X - Y) \cong F(X) - F(Y)$.
(4) F preserves pushout diagrams.

Remark 3.3. In Waldhausen's work, item 3 is subsumed by item 4. In our case since quotients (i.e. pushouts along a map to the final object) and subtraction are not the same, so we must posit an extra condition.

The work of Section 2 gives us the following.

Corollary 3.1. $\mathbf{Sch}_{/k}$ *and* $\mathbf{Var}_{/k}$ *are modified categories with cofibration.*

Proof. Axiom 1 is trivial. Axiom 2 is given to us by Schwede's results on pushouts. The first part of Axiom 3 is the standard fact that closed inclusions are closed under pullback [12]. The second part is the fact that if X, Y exist in an ambient scheme Z, with defining ideal sheaves $\mathcal{I}, \mathcal{J}$ respectively, then the pushout of X, Y along their intersection is determined by the ideal sheaf $\mathcal{I} \cap \mathcal{J}$. Thus, this pushout is a closed subscheme of X.

Axioms 4 and 5 were demonstrated in Section 2. □

Remark 3.4. Although schemes and varieties are what we care about in this paper, it is extremely likely that simplicial schemes, algebraic spaces, and stacks are all examples of modified categories with cofibrations as well. We will purse this matter further in future work.

As in Waldhausen [22], we will proceed to show that the arrow category $F_1\mathcal{C}$ of a modified category with cofibrations is also a a modified category with cofibrations. This is a first step in considering categories of filtered objects in general, i.e. sequences of cofibrations. Waldhausen [22, p.322] notes that at some point *bifiltered objects* become important and can be defined without recourse to pullbacks. That is, Waldhausen notes that a bifiltered object should be a diagram

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D \end{array}$$

where all maps are cofibrations and $\mathrm{Im}(A) \supset \mathrm{Im}(B) \cap \mathrm{Im}(C)$ and the notes that in the context of his categories with cofibration, such a definition does not make sense, and replaces it with the requirement that the pushout-product $C \amalg_A B \to D$ be a cofibration. In the context of modified categories with cofibration, such a requirement about images *does* make sense.

Definition 3.3. Given a modified category with cofibrations $\mathcal{C}$, let $F_1\mathcal{C}$ be the category of arrows $X \to Y$ in $\mathcal{C}$ with morphisms given by commuting squares.

The category $F_1\mathcal{C}$ is not automatically endowed with the structure of a modified category with cofibrations, so we need to construct such a structure for $F_1\mathcal{C}$.

Proposition 3.1. *Let* $f : (W \hookrightarrow X) \to (Y \hookrightarrow Z)$ *be a map in* $F_1\mathcal{C}$*. Then define* f *to be a cofibration if*

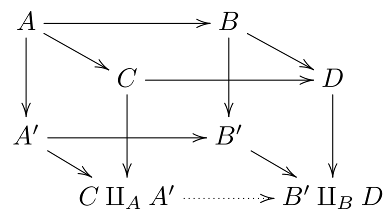
(1) $W \to Y$ *and* $X \to Z$ *are.*
(2) *If* $W \cong X \times_Z Y$*, i.e. the square is pullback.*

Denote this collection of maps as $\mathbf{co}\ F_1\mathcal{C}$*. This turns* $\mathcal{C}$ *into a modified category with cofibrations.*

Proof. First, we have to check that the cofibrations form a category. Let $(Y \hookrightarrow X) \to (Y' \hookrightarrow X') \to (Y'' \hookrightarrow X'')$ be the composition of two cofibrations. Cofibrations compose and if both squares are pullback, then the large square is pullback.

We now verify the axioms in turn.

First, pushouts along closed inclusions in both directions admit cobase change. That is, we need to show that in the diagram



where all arrows except the dotted ones are cofibrations that the dotted one is also a cofibration. This follows exactly as in Waldhausen. In particular, we write

$$A' \amalg_A D \cong (A' \amalg_A B) \amalg_B D \to B' \amalg_B D.$$

Then by cobase change and the fact that $A' \amalg_A B \to B'$ is a cofibration, this is cofibration. Further, $A' \amalg_A C \to B' \amalg_B D$ factors through $A' \amalg_A D \to B' \amalg_B D$ and so by cancellation, the dotted arrow is a cofibration.

Second, we need to check that pullbacks exist and satisfy base change. This much is simple. The fact that the pushout-product map is a cofibration follows from the above — this is equivalent to a statement about cobase change.

The fact that subtraction exists is guaranteed by the second property of the required subtraction operation.

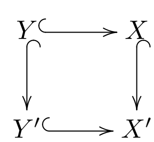
That subtraction commutes with pushout is tedious, but not difficult. □

The heavy lifting in the process of delooping in Waldhausen's setup is borne by additivity theorems. We must define the categories where such additivity statements will live in this context.

Definition 3.4. Let $F_1^+\mathcal{C}$ denote the category of pairs

$$(i : Y \hookrightarrow X, X - Y)$$

where $X - Y$ is a choice of subtraction. The morphisms given by squares



such that $Y \cong Y' \times_{X'} X$ and $X - Y \to X' - Y'$ the unique map guaranteed by the axioms 3.1.

We need various functors out of this into order to even speak reasonably about additivity.

Definition 3.5. There are three functors $s, t, q : F_1^+\mathcal{C} \to \mathcal{C}$ which on $(i : Y \hookrightarrow X, X - Y)$ are Y, X and $X - Y$ respectively. The morphisms are clear.

The following lemma is where the condition that that we have to deal with scheme theoretic intersections arise. If we don't do that, we don't have an exact functor. The condition is not needed before this.

Lemma 3.1. *$s, t, q : F_1^+\mathcal{C} \to \mathcal{C}$ are all exact.*

Proof. Only the fact that q is exact requires proof. First we check that q takes cofibrations to cofibrations. This is exactly equivalent to Prop. 3.1.

Next, we check that q preserves pushout diagrams. This is the assertion that

$$(B' - A') \amalg_{B-A} (D - C) \cong B' \amalg_B D - A' \amalg_A D.$$

This is Axiom 5. □

3.2. **SW-Categories and $\widetilde{S}_\bullet$.** We now move on to the categorical input for K-theory in our current context. The definition bears a strong resemblence to Waldhausen categories, except we are using subtraction rather than quotients.

Definition 3.6. An **SW-category** (semi-Waldhausen category) is a modified category with cofibrations equipped with a category of weak equivalences, $w\mathcal{C}$, such that

1. The isomorphisms are contained in $w\mathcal{C}$
2. Modified gluing holds: Given the diagram

$$\begin{array}{ccccc} Y & \hookleftarrow & X & \hookrightarrow & Z \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ Y' & \hookleftarrow & X' & \hookrightarrow & Z' \end{array}$$

   we have

$$Y \amalg_X Z \simeq Y' \amalg_{X'} Z'$$

3. Subtraction is respected: If we have a commuting square

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \simeq\downarrow & & \downarrow\simeq \\ X' & \hookrightarrow & Y' \end{array}$$

   then there is an induced weak equivalnce $X - Y \xrightarrow{\simeq} X' - Y'$.

Definition 3.7. A functor $F : \mathcal{C} \to \mathcal{D}$ between SW-categories is said to be **exact** if F weak equivalences and F is exact as a functor of modified categories with cofibrations.

Now, the development above proves the following.

Proposition 3.2. *With cofibrations given by closed embeddings,* $\mathbf{Var}_{/k}$ *is an SW-category.*

Definition 3.8. Given a SW-category, $\mathcal{C}$, we define $F_\bullet\mathcal{C}$ to be a simplicial set where $F_n\mathcal{C}$ is the set of sequences of arrows

$$X_0 \to X_1 \to \cdots \to X_n.$$

For $i > 0$ the face maps are given by composition. The 0th face map is given by the usual

$$(X_0 \to X_1 \to \cdots \to X_n) \mapsto (X_1 - X_0 \to X_2 - X_0 \to \cdots \to X_n - X_0)$$

Proposition 3.3. *The category $F_n\mathbf{Var}_{/k}$ can be considered as a modified category with cofibrations by defining $X \to X'$ to be a cofibration if every square in the diagram*

$$\begin{array}{ccccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & X_n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X_1' & \longrightarrow & X_1' & \longrightarrow & X_2' & \longrightarrow & \cdots & \longrightarrow & X_{n-1}' & \longrightarrow & X_n' \end{array}$$

is pullback.

Proof. This follows from the proof for F_1. □

We could proceed to define the algebraic K-theory of $\mathbf{Var}_{/k}$ using the $F_\bullet$ construction. The point is that unlike Waldhausen categories, where there is no canonical choice of quotient, there *is* a canonical choice of subtraction. Thus, it is not technically necessary to define an $S_\bullet$-construction. However, since we want to get maps into Waldhausen categories, it will make our lives significantly easier if we make the constructions as parallel as possible. We thus retain our choices.

To construct $K(\mathbf{Var}_{/k})$ we can either a singly iterated construction or "all at once" as in [6]. We will present both, as since presenting the singly iterated construction will make manifest the difficulties we face.

To cleanly state the construction we need to define a useful indexing category.

Definition 3.9. Let $[n]$ denote the ordered set $\{0, \ldots, n\}$ considered as a category, i.e. there is a map $i \to j$ if $i \leq j$. Define $\widetilde{\mathrm{Ar}}[n]$ to be the full subcategory of $[n]^{\mathrm{op}} \times [n]$ consisting of pairs (i, j) with $i \leq j$.

Example 3.1. $\widetilde{\mathrm{Ar}}[2]$ may be visualized as

$$\begin{array}{ccccc} (0,0) & \longrightarrow & (0,1) & \longrightarrow & (0,2) \\ & & \uparrow & & \uparrow \\ & & (1,1) & \longrightarrow & (1,2) \\ & & & & \uparrow \\ & & & & (2,2) \end{array}$$

and will be referred to colloquially as "flags" below.

Definition 3.10 ($\widetilde{S}_\bullet$-construction)**.** We define $\widetilde{S}_n\mathcal{C}$ to be the category of functors

$$X : \widetilde{\mathrm{Ar}}[n] \to \mathcal{C}$$

subject to the conditions

- $X_{i,i} = \emptyset$, the empty variety.
- Every $X_{i,j} \to X_{i,k}$ where $j < k$ is a cofibration.
- The sub-diagram

  $$X_{i,j} \to X_{i,k} \leftarrow X_{j,k}$$

  is a cofibration sequence.

This defines a simplicial object as follows. The face maps are

(1) $d_0 : \widetilde{S}_n\mathcal{C} \to \widetilde{S}_{n-1}\mathcal{C}$ is given by removing the first row.

(2) $d_k : \widetilde{S}_n\mathcal{C} \to \widetilde{S}_{n-1}\mathcal{C}$ is given by deleting the kth row and column and composing the remaining maps.

The degeneracy maps are given by repetition. From this it is clear that the simplicial relations hold.

We also record the following.

Proposition 3.4. *$\widetilde{S}_\bullet\mathbf{Var}_{/k}$ may be considered as a simplicial SW-category.*

Proof. Each $\widetilde{S}_n\mathbf{Var}_{/k}$ is certainly a category with cofibrations, as it is equivalent to $F_n\mathbf{Var}_{/k}$ and $F_n\mathbf{Var}_{/k}$ is such an object. □

We can finally define our *space* $K(\mathbf{Var}_{/k})$ — more machinery is needed to prove that it may be delooped.

Definition 3.11. We define the space $K(\mathbf{Var}_{/k})$ to be $\Omega|i_\bullet\widetilde{S}_\bullet\mathbf{Var}_{/k}|$, where $|-|$ denote the simplicial realization of a bisimplicial set.

Of course, the salient property of this space holds.

Proposition 3.5. $\pi_0 K(\mathbf{Var}_{/k}) = K_0(\mathbf{Var}_{/k})$.

Proof. This follows by standard methods; see, for example, [23]. For any simplicial space (or bisimplicial set) $X_\bullet$, we can compute $\pi_1|X_\bullet|$ via generators and relations:

$$\pi_1|X_\bullet| = \langle \pi_0(X_1)\rangle/(d_1(x) = d_2(x) + d_0(x)) \qquad x \in \pi_0(X_2).$$

Here our simplicial space is $X_n = |i_\bullet S_n\mathbf{Var}_{/k}|$. Therefore, $\pi_0(X_1)$ is the set of equivalence classes of varieties up to isomorphism. Also, X_2 is the set of equivalences classes of cofiber sequences. For a cofiber sequence $X \hookrightarrow Y \leftarrow X - Y$, call it c, $d_0(c) = Y - X$, $d_1(c) = Y$ and $d_2(c) = X$. Therefore, the relations are

$$[Y] = [X] + [Y - X].$$

□

We now produce $K(\mathbf{Var}_{/k})$ as a symmetric spectrum by iterating the $\widetilde{S}_\bullet$ construction in an appropriate way; that is, the following is what one gets if we consider $\widetilde{S}_\bullet\mathcal{C}$ as an SW-category and iterate the $\widetilde{S}_\bullet$-construction. We will show in the subsequent section that this is a quasi-fibrant symmetric spectrum [13, 16].

Definition 3.12. Let $\mathcal{C}$ be an SW-category. We consider the category of functors

$$F : \widetilde{\mathrm{Ar}}[n_1] \times \cdots \times \widetilde{\mathrm{Ar}}[n_k] \to \mathcal{C}.$$

and write each object of $\widetilde{\mathrm{Ar}}[n_\ell]$ as (i_ℓ, j_ℓ). Let $S^k_{n_1,\ldots,n_k}\mathcal{C}$ be the full subcategory consisting of functors F such that

(1) $F((i_1, j_1), \ldots, (i_k, j_k)) = *$ whenever $i_\ell = j_\ell$ for some ℓ.
(2) The subfunctor $F((0, i_1), \ldots, (0, i_k)) : [n_1] \times \cdots \times [n_k] \to \mathcal{C}$ defines a cube such that every sub-face is pullback.
(3) Given $((i_1, j_1), \ldots, (i_k, j_k))$ in $\widetilde{\mathrm{Ar}}[n_1] \times \cdots \times \widetilde{\mathrm{Ar}}[n_k]$ and $1 \le \ell \le k$ and every $j_\ell \le m \le n_\ell$ the sequence

$$\begin{array}{ccc} F((i_1,j_1),\ldots,(i_\ell,j_\ell)) & \longrightarrow & F((i_1,j_1),\ldots,(i_\ell,m),\ldots,(i_k,j_k)) \\ & & \uparrow \\ & & F((i_1,j_1),\ldots,(j_\ell,m),\ldots,(i_k,j_k)) \end{array}$$

Using this we may define the symmetric spectrum $K(\mathcal{C})$

Definition 3.13. Let $\mathcal{C}$ be an SW-category and define

$$K(\mathcal{C})(k) = |N_\bullet(wS^{(k)}_{\bullet,\dots,\bullet}\mathcal{C})|.$$

This space has a Σ_k-action given by permuting the simplicial directions.

We introduce one more generalization in order to more cleanly define products later on; this simply language, and we are not introducing any new notions. We will follow Geisser-Hesselholt [10] in defining products, and so follow them in defining an $S_\bullet$-construction appropriate to the task. The only modification is to consider $S^Q\mathcal{C}$ where Q is a finite set. That is, instead of indexing on numbers, we index on finite sets. This serves to make the action by the symmetric group more transparent.

Definition 3.14. Let Q be a finite set. Consider positive integers n_i indexed on Q, i.e. where $i \in Q$. Then $S^Q_{n_1,\dots,n_{|Q|}}\mathcal{C}$ is a functor from the arrow category

$$F : \widetilde{\mathrm{Ar}}[n_1] \times \cdots \times \widetilde{\mathrm{Ar}}[n_{|Q|}] \to \mathcal{C}$$

satisfying the same requirements as above.

3.3. **Additivity.** Additivity is the defining property of K-theory [3, 2] and almost every other standard property of K-theory follows from additivity [21]. For our purposes, additivity will only be used to produce fibration sequences that produce deloopings — this is because the category of varieties has very few additional properties that are needed to produce the other K-theory theorems (e.g. cylinder functors).

Definition 3.15. Let $E(\mathcal{C})$ be the category $F_1^+\mathcal{C}$ introduced earlier. We will continue to use the notation $s, t, q : E(\mathcal{C}) \to \mathcal{C}$ from Defn. 3.5.

Theorem 3.1 (Additivity). *Let $\mathcal{C}$ be an SW-category. Consider the map*

$$A = (s, q) : E(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}.$$

Upon applying $\widetilde{S}_\bullet$ we get an equivalence of simplicial sets

$$\widetilde{S}_\bullet E(\mathcal{C}) \to \widetilde{S}_\bullet\mathcal{C} \times \widetilde{S}_\bullet\mathcal{C}.$$

The slickest, most powerful proof of additivity is due to McCarthy [17]. We will review the proof here in order to convince ourselves that SW-categories have sufficiently nice properties to allow McCarthy's proof to go through. The key point is that while pushouts are used extensively in the proof, *only* pushouts where *both* legs are cofibrations are needed. As pointed out in Section 2, these are exactly the types of pushouts that we do have. Again, we emphasize that although some minor details differ, the proof of additivity is due to McCarthy. We are simply showing that with some tweaks the proof goes through for a slightly different class of categories.

Definition 3.16. Let $\mathcal{C}, \mathcal{D}$ be modified categories with cofibrations and let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor. Define a bisimplicial set $\mathcal{C} \otimes_{S_\bullet f} \mathcal{D}$ by defining

$$(\mathcal{C} \otimes_{S_\bullet f} \mathcal{D})([m], [n])$$

to be diagrams (we are omitting the rows below the first)

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_m$$

$$Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots \hookrightarrow Y_m \hookrightarrow \cdots \hookrightarrow Y_{m+n}$$

such that $f(X_i) = Y_i$ and $f(X_i \to X_{i+i}) = Y_i \to Y_{i+1}$. The face and degeneracy maps are given by composition and repetition, respectively.

Definition 3.17. Given a simplicial set X let X^L denote the bisimplicial set defined by $X^L([m],[n]) = X([m])$ and similarly let X^R denote the bisimplicial set $X^R([m],[n]) = X([n])$.

The categories in Def. 3.16 are introduced in order to state the following proposition.

Proposition 3.6 (McCarthy)**.** *The following are equivalent*

(1) $\widetilde{S}_\bullet f : \widetilde{S}_\bullet \mathcal{C} \to \widetilde{S}_\bullet \mathcal{D}$ *is a homotopy equivalence*
(2) *The bisimplicial map*

$$\mathcal{C} \otimes_{\widetilde{S}_\bullet f} \mathcal{D} \to \widetilde{S}_\bullet \mathcal{D}^R$$

is a homotopy equivalence.

Remark 3.5. This proposition uses nothing about categories with cofibrations, or weak equivalences, or any other Waldhausen-type structure.

Now define a self-map

$$E_n : (\mathcal{C} \otimes_{S_\bullet F} \mathcal{D})(-,[n]) \to (\mathcal{C} \otimes_{S_\bullet F} \mathcal{D})(-,[n])$$

via a quotienting procedure. We take the standard diagrams to (again, omitting the rows below the first)

$$0 = \cdots = 0$$

$$0 = \cdots = 0 = X_0 - X_0 \hookrightarrow X_1 - X_0 \hookrightarrow \cdots \hookrightarrow X_n - X_0$$

The above proposition implies

Corollary 3.2. *If E_n are homotopy equivalences for all n, then $\widetilde{S}_\bullet F : \widetilde{S}_\bullet \mathcal{C} \to \widetilde{S}_\bullet \mathcal{D}$ is.*

Remark 3.6. Again, this relies on nothing except the ability to take complements (or in the case of categories with cofibration, quotients).

Let $A : E(\mathcal{C}) \xrightarrow{(s,t)} \mathcal{C} \times \mathcal{C}$ be the functor defined by the additivity functors (Defn. 3.5). McCarthy considers diagrams

$$E(\mathcal{C}) \otimes_{S_\bullet A} \mathcal{C}^2.$$

It is unfortunate, but at some point we will need to visualize diagrams like this. Here is their typical form (as always omitting the rows after the first)

$$\begin{array}{ccccccccc}
\emptyset & = & A_0 & \hookrightarrow & A_1 & \hookrightarrow & \cdots & \hookrightarrow & A_m \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\emptyset & = & C_0 & \hookrightarrow & C_1 & \hookrightarrow & \cdots & \hookrightarrow & C_m \\
 & & \uparrow{\scriptstyle \mathrm{op}} & & \uparrow{\scriptstyle \mathrm{op}} & & & & \uparrow{\scriptstyle \mathrm{op}} \\
\emptyset & = & B_0 & \hookrightarrow & B_1 & \hookrightarrow & \cdots & \hookrightarrow & B_m
\end{array}$$

$$\emptyset = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_m \hookrightarrow S_0 \hookrightarrow \cdots \hookrightarrow S_n$$

$$\emptyset = B_0 \hookrightarrow B_1 \hookrightarrow \cdots \hookrightarrow B_m \hookrightarrow T_0 \hookrightarrow \cdots \hookrightarrow T_n$$

Remark 3.7. We note that the only difference between these diagrams and the diagrams that appear in [17] is the fact that the arrows between the Bs and Cs go in opposite directions.

We will now show that E_n for the functor $(s,q) : E(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is a homotopy equivalence. McCarthy does this by defining a map Γ and showing that Γ is homotopic to the identity and that E_n restricted to a certain subspace is homotopic to the identity on that subspace.

Definition 3.18. The map $\Gamma : E(\mathcal{C}) \times_{S_\bullet A} \mathcal{C}^2(-,[n]) \to E(\mathcal{C}) \times_{S_\bullet A} \mathcal{C}^2(-,[n])$ is defined by taking diagrams as above to diagrams (as always, omitting rows below the first)

$$\begin{array}{ccccccccc}
\emptyset & = & \emptyset & = & \emptyset & = & \cdots & = & \emptyset \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
\emptyset & = & B_0 & \hookrightarrow & B_1 & \hookrightarrow & \cdots & \hookrightarrow & B_m \\
 & & \| & & \| & & & & \| \\
\emptyset & = & B_0 & \hookrightarrow & B_1 & \hookrightarrow & \cdots & \hookrightarrow & B_m
\end{array}$$

$$\emptyset = \emptyset = \emptyset = \cdots = \emptyset = S_0 - S_0 \hookrightarrow S_1 - S_0 \hookrightarrow \cdots \hookrightarrow S_m - S_0$$

$$\emptyset = B_0 \hookrightarrow B_1 \hookrightarrow \cdots \hookrightarrow B_m \hookrightarrow T_0 \hookrightarrow T_1 \hookrightarrow \cdots \hookrightarrow T_n$$

Definition 3.19. Let the subspace $\mathcal{X} \subset E(\mathcal{C}) \times_{S_\bullet A} \mathcal{C}^2(-,[n])$ denote the subspace where all of the A_i are $\emptyset$.

Now, $E_n|_{\mathcal{X}} \simeq \mathrm{Id}\,|_{\mathcal{X}}$ and we note that Γ is a retraction of $E(\mathcal{C}) \times_{S_\bullet A} \mathcal{C}^2(-,[n])$ onto $\mathcal{X}$. To complete the proof that E_n is a homotopy equivalence, and thus additivity, we just need to show that Γ is homotopic to the identity.

This is done by producing an explicit homotopy

$$h : (E(\mathcal{C}) \times_{S_\bullet A} \mathcal{C}^2(-,[n])) \times \Delta^1 \to (E(\mathcal{C}) \times_{S_\bullet A} \mathcal{C}^2(-,[n]))$$

Let $e \in E(\mathcal{C}) \otimes_{S_\bullet A} \mathcal{C}^2([m],[n])$ and let h_i be the restriction of h to the ith vertex. We visually define $h_i(e)$ to be

$$\begin{array}{ccccccccccc}
0 = A_0 & \hookrightarrow & A_1 & \hookrightarrow \cdots & A_i & \hookrightarrow & S_0 & = & S_0 & = \cdots = & S_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_0 & \hookrightarrow & C_1 & \hookrightarrow \cdots & C_i & \hookrightarrow & C_i \amalg_{A_i} S_0 & \hookrightarrow & C_{i+1} \amalg_{A_{i+1}} S_0 & \cdots \hookrightarrow & C_m \amalg_{A_m} S_0 \\
\uparrow^{\mathrm{op}} & & \uparrow^{\mathrm{op}} & & \uparrow^{\mathrm{op}} & & \uparrow^{\mathrm{op}} & & \uparrow^{\mathrm{op}} & & \uparrow^{\mathrm{op}} \\
B_0 & \hookrightarrow & B_1 & \hookrightarrow \cdots & B_i & = & B_i & \hookrightarrow & B_{i+1} & \hookrightarrow \cdots \hookrightarrow & B_m
\end{array}$$

$$0 = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_i \hookrightarrow S_0 = S_0 = \cdots = S_0$$

$$0 = B_0 \hookrightarrow B_1 \hookrightarrow \cdots \hookrightarrow B_i = B_i \hookrightarrow B_{i+1} \hookrightarrow \cdots \hookrightarrow B_m$$

Note that here we are using the existence of pushouts provided by Th. 2.2. This is one of the critical points where that fact is used.

Although we are not displaying the levels below the upper row the the diagrams above, we will need to reference the rows below. We'll refer to the three rows as the A, B, C rows. The choices of subtraction in the diagram e will be refered to by $A_{k,l}$, $B_{k,l}$ and $C_{k,l}$. The choices in the diagram $h_i(e)$ will be referred to more inelegantly as $h_i(e)^A_{k,l}$ and similarly. We now explicitly identify the quotient rows. For $i \geq 0$ define

$$h_i(e)^A_{k,l} = \begin{cases} A_{k,l} & k, l \leq i \\ S_0 - A_{0,k} & k \leq i, l > i \\ \emptyset & \text{otherwise} \end{cases}$$

$$h_i(e)^C_{k,l} = \begin{cases} C_{k,l} & k, l \leq i \\ C_{k,l-1} \amalg_{A_{k,l-1}} h_i(e)^A_{k,l} & k \leq i, l > i \\ B_{k,l} & l, k > i \end{cases}$$

$$h_i(e)^B_{k,l} = \begin{cases} B_{k,l} & k, l \leq i \\ B_{k,i} & l = i+1, k = i+1, l \neq k \\ \emptyset & l = k = i+1 \\ B_{k,l-1} & \text{otherwise} \end{cases}$$

The appendix depicts a few of these diagrams written out.

There are a few things to check. One is that this is in fact a simplicial homotopy. The other is that this corresponds to what we think should be the simplicial homotopy! That is, we have defined this homotopy only in terms of pushouts. We need to check that these pushouts model subtraction. Further, we should be careful about the agreement of certain maps in the diagram.

For the most part, the maps in the diagram are clear. One that requires comment is the map in $h_i^C(e)$ from $B_{k,l}$ to $C_{k,l} \amalg_{A_{k,l}} (S_0 - A_k)$. This will be the composition

$$B_{k,l} \xrightarrow{\cong} C_{k,l} - A_{k,l} \xrightarrow{\cong} C_{k,l} \amalg_{A_{k,l}} (S_0 - A_k) - (S_0 - A_k) \hookrightarrow C_{k,l} \amalg_{A_{k,l}} (S_0 - A_k)$$

Each of these isomorphisms and inclusions is uniquely determined by data in e. The other maps that require comment are those from $h^B_{k,l}$ to $h^C_{k,l}$ — whenever both of them are $B_{k,l}$ the map between them will be the identity.

The following proposition verifies that the above diagram models the appropriate subtractions.

Proposition 3.7. *We have the following isomorphisms*

$$h_i^C(e)^C_{0,l} - C_{0,1} \cong h_i^C(e)^C_{1,l}$$
$$h_i^C(e)^C_{0,l} - h_i^C(e)_{0,k} \cong B_{k,l-1} \qquad l > i$$

Proof. For the first item, the object on the left is

$$C_{0,l} \amalg_{A_{0,l}} S_0 - C_{0,1}.$$

The diagram

$$\begin{array}{ccccc} C_{0,1} & \longleftarrow & A_{0,1} & \longrightarrow & A_{0,1} \\ \downarrow & & \downarrow & & \downarrow \\ C_{0,l} & \longleftarrow & A_{0,l} & \longrightarrow & S_0 \end{array}$$

satisfies the properties of Defn. 3.1 Axiom 5 and gives the result. Note that all of the maps in the diagram are determined by the maps in e and so the isomorphism is in fact given by data contained in e.

The second is verified in an entirely analogous manner. That is $h_i^C(e)_{0,l} \cong C_{0,l-1} \amalg_{A_{0,l-1}} S_0$ and $h_i^C(e)_{0,k} \cong C_{0,k-1} \amalg_{A_{0,k-1}} S_0$. Then note that

$$C_{0,l-1} \amalg_{A_{0,l-1}} S_0 - S_0 \cong B_{0,l-1}$$
$$C_{0,k-1} \amalg_{A_{0,k-1}} S_0 - S_0 \cong B_{0,k}$$

and subtract. Note again that all maps are determined by e. □

Thus the above diagram, though it involves only pushouts, sufficiently models our situation.

We will now verify that h_i is a simplicial homotopy. Recall that this means that the following identities hold:

$$\begin{cases} d_0 h_0 = \Gamma \\ d_{q+1} h_q = \mathrm{Id} \end{cases} \qquad \begin{cases} d_i h_j = h_{j-1} d_i & i < j \\ d_{j+1} h_{j+1} = d_{j+1} h_j & \\ d_i h_j = h_j d_{i-1} & i > j+1 \end{cases} \qquad \begin{cases} s_i h_j = h_{j+1} s_i & i \le j \\ s_i h_j = h_j s_{i-1} & i > j \end{cases}$$

First, it is clear that $d_{q+1} h_q = \mathrm{Id}$. It is also clear that $d_0 h_0 = \Gamma$.

The identities involving degeneracy hold trivially.

The middle group of identities is not hard:

$\boxed{d_i h_j = h_{j-1} d_i}$ when $i < j$. This part only involves the $C_{k,l}$ and thus holds by the simplicial identities in the $C_{k,l}$ part of e.

$\boxed{d_{j+1} h_{j+1} = d_{j+1} h_j}$. This identity is clear from the definitions.

$\boxed{d_i h_j = h_j d_{i-1}}$. Again, this is not difficult. The identity comes from the simplicial identities on the $B_{k,l}$ part of e and the fact that pushouts are chosen functorially and based on maps in e. (See the appendix for a useful picture).

With these verifications we know that h_i is a simplicial homotopy and this ends the proof of the additivity theorem.

3.4. **Delooping.** Of course, additivity is a stepping stone to delooping for us. That is, additivity will allow us to show that the adjoint to $\Sigma K(\mathcal{C})(k) \to K(\mathcal{C})(k+1)$ is a weak equivalence, which exhibits $K(\mathcal{C})(1)$ as an infinite loop space. This will also allow us to identify $K(\mathcal{C})$ as a quasi-fibrant symmetric spectrum.

We will approach delooping as Waldhausen does. However, we need a definition first.

Definition 3.20. Let $PX_\bullet$ denote the simplicial path space of the simplicial set $X_\bullet$. Then for SW-categories $\mathcal{A}$ and $\mathcal{B}$ with an exact functor $f : \mathcal{A} \to \mathcal{B}$ we define $\widetilde{S}_n(f : \mathcal{A} \to \mathcal{B})$ via pullback:

$$\begin{array}{ccc} \widetilde{S}_n(f : \mathcal{A} \to \mathcal{B}) & \longrightarrow & (P\widetilde{S}_\bullet\mathcal{B})_{n+1} \\ \downarrow & & \downarrow \\ \widetilde{S}_n\mathcal{A} & \longrightarrow & \widetilde{S}_n\mathcal{B} \end{array}$$

Additivity gets used int the proof of the following proposition.

Proposition 3.8 (cf. [22])**.** *Let $\mathcal{A}, \mathcal{B}$ be modified categories with cofibration (and weak equivalence which are isomorphisms). Suppose $f : \mathcal{A} \to \mathcal{B}$ is exact. Then*

$$i\widetilde{S}_\bullet\mathcal{B} \to i\widetilde{S}_\bullet\widetilde{S}_\bullet(f : \mathcal{A} \to \mathcal{B}) \to i\widetilde{S}_\bullet\widetilde{S}_\bullet\mathcal{A}$$

is a fibration up to homotopy.

Proof. As in Waldhausen. The proof doesn't use anything special about categories with weak equivalences as defined in Waldhausen. It simply uses additivity. □

We use this to obtain:

Corollary 3.3. *The sequence*

$$i\widetilde{S}_\bullet\mathcal{C} \to P(i\widetilde{S}_\bullet\widetilde{S}_\bullet\mathcal{C}) \to i\widetilde{S}_\bullet\widetilde{S}_\bullet\mathcal{C}$$

is a fibration sequence up to homotopy, i.e.

$$|iS_\bullet\mathcal{C}| \simeq \Omega|iS_\bullet S_\bullet\mathcal{C}|$$

Corollary 3.4. *We can define an infinite loop space, and thus a connective spectrum $K(\mathbf{Var}_{/k})$.*

In fact, as the next section shows, we can do even better.

3.5. **Multiplicative Structure.** There is more structure to the category $\mathbf{Var}_{/k}$ than we have used thus far, in particular, there is a cartesian product: given k-varieties X, Y we can consider $X \times_k Y$. This much is used to produce the ring structure on $K_0(\mathbf{Var}_{/k})$. We can also use it to produce a homotopy-coherent product structure on $K(\mathbf{Var}_{/k})$, that is an E_∞-ring structure.

Now, let $\mathcal{C}$ be a strict symmetric monoidal SW-category. We would like to induce a multiplicative structure on the spectrum $K(\mathcal{C})$. In order for this to be the case we need a condition on the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Definition 3.21. Let $\mathcal{C}$ be an SW-category. Then $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is **biexact** if

(1) $X \times \emptyset$ and $\emptyset \times X$ are both $\emptyset$
(2) $X \times (-)$ and $(-) \times X$ are exact functors

(3) For $X \to Y$ and $X' \to Y'$ cofibrations the pushout-product

$$X' \times Y \amalg_{X \times Y} X \times Y' \to X' \times Y'$$

is a cofibration.

Remark 3.8. Given a symmetric monoidal category $\mathcal{C}$ it can always be rigidified into a permutative category (or what is sometimes called a strict symmetric monoidal category). The input for generating commutative ring spectrum will need to be permutative categories.

Definition 3.22. Let $\mathcal{C}$ be a permutative category. There is an induced product

$$S_\bullet^Q \mathcal{C} \times S_\bullet^{Q'} \mathcal{C} \to S_\bullet^{Q \amalg Q'} \mathcal{C}$$

given by amalgamating the morphisms in the arrow categories. This gives a $\Sigma_m \times \Sigma_n$-equivariant map

$$K(\mathcal{C})_m \times K(\mathcal{C})_n \to K(\mathcal{C})_{m+n}$$

which descends to

$$K(\mathcal{C})_m \wedge K(\mathcal{C})_n \to K(\mathcal{C})_{m+n}.$$

Theorem 3.2. [5, Th 2.8][10] *Let $\mathcal{C}$ be a symmetric monoidal SW-category with $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ biexact. Then $K(\mathcal{C})$ is an E_∞-ring spectrum.*

Of course, we would like this result for $\mathcal{C} = \mathbf{Var}_{/k}$. That means that we have to show that the cartesian product is biexact. This is the content of the proposition below.

Proposition 3.9. *Let $X \hookrightarrow X'$ and $Y \hookrightarrow Y'$ be cofibrations of varieties. Then the pushout-product*

$$X \times Y' \amalg_{X \times Y} X' \times Y \to X' \times Y'$$

is a cofibration.

There is an axiomatic proof and hands-on proof.

Proof 1. The diagram

$$\begin{array}{ccc} X \times Y & \longrightarrow & X' \times Y \\ \downarrow & & \downarrow \\ X \times Y & \longrightarrow & X' \times Y' \end{array}$$

is pullback. Since we have verified the axioms for $\mathbf{Var}_{/k}$, this means the pushout-product of this diagram is a cofibration. □

Proof 2. Closed inclusions are obviously closed under products, so this is a valid pushout. As usual, the question is local. So we consider $\operatorname{Spec} A/I \to \operatorname{Spec} A$ and $\operatorname{Spec} B/I \to \operatorname{Spec} B$. The pushout we're dealing with is then

$$\operatorname{Spec}(A/I \otimes_k B) \coprod_{\operatorname{Spec}(A/I \otimes_k B/J)} \operatorname{Spec}(A \otimes_k B/I)$$

However, this is spec of a fiber product

$$\operatorname{Spec}\big((A/I \otimes_k B) \times_{A/I \otimes_k B/J} A \otimes_k B/J\big)$$

and this is in turn isomorphic to $\operatorname{Spec}(A \otimes B)$. This is thus locally an isomorphism, and so a cofibration. □

We now have

Corollary 3.5. *The usual product induces a paing* $\mathbf{Var}_{/k} \times \mathbf{Var}_{/k} \to \mathbf{Var}_{/k}$ *which descends to a product on* $K(\mathbf{Var}_{/k})$*. Thus,* $K(\mathbf{Var}_{/k})$ *is an* E_∞*-ring spectrum.*

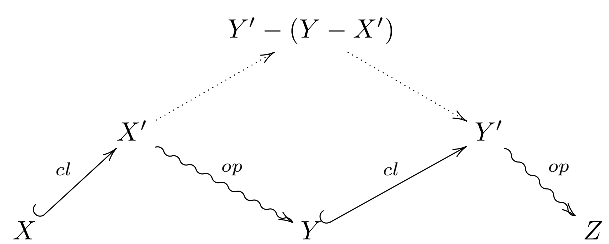
4. $\widetilde{Q}$-construction, Duality, Functoriality

Up to this point we have been using an $S_\bullet$ construction to define K-theory. This construction has some virtues, in particular making the delooping more readily apparent. However, it will become convenient to use a construction similar to Quillen's Q-construction. In particular, introducing the Q-construction will allow us to prove devissage in the case of the category $\mathbf{Var}_{/k}$ (Thm. 5.2).

Typically the Q-construction is applied to exact categories. The category $\mathbf{Var}_{/k}$ is emphatically not exact or additive or pre-additive. However, there is just enough structure to define a sort of span category and compositions.

Definition 4.1. Let $\widetilde{Q}\mathbf{Var}_{/k}$ be the category with

- **objects**: Varieties over k
- **morphisms**: Compositions of two morphisms of varieties, where the first is closed and the second is open. That is, a typical morphism between C and Y looks like
$$C \overset{cl}{\hookrightarrow} X \overset{\text{op}}{\hookrightarrow} Y$$
- **composition**: Given morphisms $X \to Y$ and $Y \to Z$ whicn are presented as
$$X \xrightarrow{cl} X' \xrightarrow{op} Y \qquad Y \xrightarrow{cl} Y' \xrightarrow{op} Z$$
we draw the usual span diagram



We need to check that morphisms in fact compose. We note that since $X' \to Y$ is open $(Y - X')$ is closed in Y, and thus Y' since closed embeddings compose. Thus, $Y' - (Y - X')$ is open in Y'. Furthermore, considering $X \to Y' - (Y - X')$ we see that $Y' - (Y - X') - X' \cong Y' - Y$ which is open in Y'. Now, $Y \cap (Y' - (Y - X')) \cong X'$ and this intersection is closed in $Y' - (Y - X')$, so X' is closed in $Y' - (Y - X')$.

So, the composition exists and is the morphism
$$X \xrightarrow{cl} Y' - (Y - X) \xrightarrow{op} Z$$
in $\widetilde{Q}\mathbf{Var}_{/k}$.

As in the case of Q and $S_\bullet$, there is a fairly direct comparison between the constructions. The following standard definition will be needed in the proof of the comparison.

Definition 4.2. Let Δ be the usual simplicial category. We define a functor, subdivision, $\mathrm{sd} : \Delta \to \Delta$. Let $\mathrm{sd}[n] = [2n+1]$ and $\mathrm{sd}(f) = f \amalg f$. Furthermore, for X a simplicial set, let $\mathrm{sd}\, X$ denote the pullback by subdivision.

The following is well known (see, for example,[8]).

Proposition 4.1. *On geometric realization* $|X_\bullet| \cong |\,\mathrm{sd}\, X_\bullet|$.

Theorem 4.1 ($\widetilde{Q} = \widetilde{S}_\bullet$). *We have*

$$N(\widetilde{Q}\mathbf{Var}_{/k}) \simeq N(i\widetilde{S}_\bullet\mathbf{Var}_{/k})$$

Proof. This is proved exactly the same way as in Waldhausen. The point is to take a subdivision and note that $\mathrm{sd}\, i\widetilde{S}_\bullet\mathbf{Var}_{/k}$ is equivalent to $N\widetilde{Q}\mathbf{Var}_{/k}$. To see that this is true, we give an example in this situation, as Waldhuasen does. We have

$$\mathrm{sd}[3] = [3] \star [3] = \underline{3} < \underline{2} < \underline{1} < 0 < \overline{0} < \overline{1} < \overline{2} < \overline{3}.$$

An object in $\mathrm{sd}\, i\widetilde{S}_3\mathbf{Var}_{/k}$ is thus represented as

$$X_{\underline{3},\underline{2}} \hookrightarrow X_{\underline{3},\underline{1}} \hookrightarrow X_{\underline{3},\underline{0}} \hookrightarrow X_{\underline{3},\overline{0}} \hookrightarrow X_{\underline{3},\overline{1}} \hookrightarrow X_{\underline{3},\overline{2}} \hookrightarrow X_{\underline{3},\overline{3}}$$

and we define

$$X_{i,j} = X_{\underline{3},j} - X_{\underline{3},i}$$

and the associate the sequence of cofibrations above to the diagram

$$\begin{array}{ccccccc}
X_{\underline{3},\overline{0}} & \xrightarrow{cl} & X_{\underline{3},\overline{1}} & \xrightarrow{cl} & X_{\underline{3},\overline{2}} & \xrightarrow{cl} & X_{\underline{3},\overline{3}} \\
\uparrow{\scriptstyle op} & & \uparrow{\scriptstyle op} & & \uparrow{\scriptstyle op} & & \\
X_{\underline{3},\overline{0}} - X_{\underline{3},\underline{2}} & \xrightarrow{cl} & X_{\underline{3},\overline{1}} - X_{\underline{3},\underline{2}} & \xrightarrow{cl} & X_{\underline{3},\overline{2}} - X_{\underline{3},\underline{2}} & & \\
\uparrow{\scriptstyle op} & & \uparrow{\scriptstyle op} & & & & \\
X_{\underline{3},\overline{0}} - X_{\underline{3},\underline{1}} & \xrightarrow{cl} & X_{\underline{3},\overline{1}} - X_{\underline{3},\underline{1}} & & & & \\
\uparrow{\scriptstyle op} & & & & & & \\
X_{\underline{3},\overline{0}} - X_{\underline{3},\underline{0}} & & & & & &
\end{array}$$

where all of the upward arrows are open embeddings and all of the horizontal arrows are closed embeddings. This diagram is then the composition of three morphisms in $\widetilde{Q}\mathbf{Var}_{/k}$. □

We thus have another way of dealing with the K-theory of varieties. This will be useful for proving K-theory theorems.

Definition 4.3. By the above theorem, we can equally well define

$$K(\mathbf{Var}_{/k}) = \Omega|N\widetilde{Q}\mathbf{Var}_{/k}|$$

4.1. **Duality.** At the beginning of the paper we chose the cofibrations of our theory to be closed inclusions. This was a somewhat arbitrary choice, and was simply motiviated by the desire to front-load the hard work necessary to prove the existence of various pushouts. The other perfectly reasonable choice for cofibrations is, of course, open inclusions.

To this end, we may make the following definition.

Definition 4.4. Let $\mathbf{Var}^{\text{open}}_{/k}$ be the SW-category with objects k-varieties, morphsims maps of varieties, cofibrations open inclusions and weak equivalences isomorphisms.

Of course, we have not shown that the various pushouts that we need to exist for $\mathbf{Var}^{\text{op}}_{/k}$ actually exist. However, these statements are dual to the statements proved in Section 1.

We may now state and prove the following.

Theorem 4.2. *Let $\mathbf{Var}^{open}_{/k}$ and $\mathbf{Var}^{closed}_{/k}$ be the modified Walhdausen categories where the cofibrations are open immersions and closed immersions, respectively. Then*

$$K(\mathbf{Var}^{open}_{/k}) \simeq K(\mathbf{Var}^{closed}_{/k}).$$

Proof. We consider flags in $\widetilde{S}_\bullet \mathbf{Var}^{\text{closed}}_{/k}$. For example, elements of $\widetilde{S}_n \mathbf{Var}_{/k}$ will take the form below:

$$\begin{array}{ccccccccc}
X_0 & \xrightarrow{cl} & X_1 & \xrightarrow{cl} & X_2 & \xrightarrow{cl} & \cdots & \xrightarrow{cl} & X_n \\
 & & \uparrow{\scriptstyle op} & & \uparrow{\scriptstyle op} & & & & \uparrow{\scriptstyle op} \\
 & & X_{01} & \xrightarrow{cl} & X_{02} & \xrightarrow{cl} & \cdots & \xrightarrow{cl} & X_{0n} \\
 & & & & \uparrow{\scriptstyle op} & & & & \uparrow{\scriptstyle op} \\
 & & & & X_{12} & \xrightarrow{cl} & \cdots & \xrightarrow{cl} & X_{1n} \\
 & & & & & & & & \uparrow{\scriptstyle op} \\
 & & & & & & & & \vdots \\
 & & & & & & & & \uparrow{\scriptstyle op} \\
 & & & & & & & & X_{n,n-1}
\end{array}$$

All horizontal arrows are closed inclusions, while all vertical arrows are open inclusions. The flags in $\widetilde{S}_\bullet \mathbf{Var}^{\text{closed}}_{/k}$ are then in exact corresponence with the flags in $\widetilde{S}_\bullet \mathbf{Var}^{\text{open}}_{/k}$: it is clear that a flag in one will determine a flag in the other. Furthermore, the simplicial structure is exactly the same in each case. □

Remark 4.1. A similar duality statement will hold for the $\widetilde{Q}$-construction where instead of declaring the morphisms in $\widetilde{Q}$ that are closed inclusions followed by open inclusions, we have the reverse. However, we will not need such a statement in the sequel.

Remark 4.2. Such a statement is quite similar to statements about Waldhausen K-theory in exact categories, where one may use either fibrations or cofibrations to produce K-theory. [4, 1].

4.2. **Functoriality.** Thus far we have considered varieties over a field k. Of course, one of the beautiful things about Grothendieck's philosophy of algebraic geometry is that everything may be done relatively. There is no obstruction to considering the category of varieties over another fixed variety, X. Let us denote this category by $\mathbf{Var}_{/X}$. It is perfectly sensible to define $K_0(\mathbf{Var}_{/X})$ in the same way we defined it over fields. The following theorem comes for (almost) free from the work above.

Theorem 4.3. *There is a E_∞-ring spectrum $K(\mathbf{Var}_{/X})$ such that*

$$\pi_0 K(\mathbf{Var}_{/X}) \cong K_0(\mathbf{Var}_{/X}).$$

Proof. First, it's clear one can define pushouts in $\mathbf{Var}_{/X}$: we define the pushouts without reference to the base variety, and maps from the pushout to the base variety come from universal properties.

The product structure is given by the bifunctor $(-)\times_X(-)$. Since pullbacks preserve both open and closed inclusions, this is biexact. Thus, this product structure descends to an E_∞-structure on K-theory. $\square$

We can now state the following

Definition 4.5. Let $f : X \to X'$ be a map of varieties. There is a functor $\mathbf{Var}_{/X} \to \mathbf{Var}_{/X'}$ given by post-composition with f. This is obviously exact and gives a map of spectra

$$f_* : K(\mathbf{Var}_{/X}) \to K(\mathbf{Var}_{/X'}).$$

Similarly, there is a functor $\mathbf{Var}_{/X'} \to \mathbf{Var}_{/X}$ given by pulling back. Pulling back is an exact functor and so we may define a map of spectra

$$f^* : K(\mathbf{Var}_{/X'}) \to K(\mathbf{Var}_{/X'})$$

Remark 4.3. In subsequent work we will develop a six-functor formalism of such objects.

5. Devissage

Algebraic K-theory of categories can be thought of as an invariant that tells us how objects of the category are "broken up." If all elements of a category $\mathcal{C}$ can be broken into pieces lying in a subcategory $\mathcal{A} \subset \mathcal{C}$, then one would hope $K(\mathcal{C}) \simeq K(\mathcal{A})$. In the case of abelian categories, the precise statement of this is Quillen's devissage theorem.

Theorem 5.1 (Quillen)**.** *Suppose $\mathcal{A} \subset \mathcal{C}$ is an inclusion of abelian categories where $\mathcal{A}, \mathcal{C}$ are exact and $\mathcal{A}$ is closed under subobjects and quotients. Suppose furthemore that every $C \in \mathcal{C}$ has a finite filtration*

$$0 = C_n \subset C_{n-1} \subset \cdots \subset C_1 \subset C_0 = C$$

such that $C_{i-1}/C_i \in \mathcal{A}$. Then there is a homotopy equivalence

$$K(\mathcal{A}) \simeq K(\mathcal{C}).$$

The category $\mathbf{Var}_{/k}$ is decidely not abelian. In fact, SW-categories have very few appealing properties from a homotopy theoretic standpoint. For example, $\mathbf{Var}_{/k}$ lacks any kind of cylinder functor, lacks almost all colimits, etc. However, it does have filtration properties: non-smooth varieties can be split up into sums of smooth varieties. That is, given a variety X, there is a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

such that X_0 is smooth and $X_i - X_{i-1}$ is smooth for all i. In particular this proves a version of devissage for $K_0(\mathbf{Var}_{/k})$.

Proposition 5.1. *Let $\mathbf{Var}^{sm}_{/k}$ be the full subcategory of $\mathbf{Var}_{/k}$ consisting of smooth varieties. Then*

$$K_0(\mathbf{Var}^{sm}_{/k}) \cong K_0(\mathbf{Var}_{/k})$$

Proof. We write the filtration

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

where all inclusions are closed. Then

$$[X] = [X_n] = [X_n - X_{n-1}] + [X_{n-1} - X_{n-2}] + \cdots [X_1 - X_0] + [X_0]$$

so the inclusion $K_0(\mathbf{Var}^{\mathrm{sm}}_{/k}) \to K_0(\mathbf{Var}_{/k})$ is bijective. Further, it is certainly a ring homomoprhism. □

Of course, since this is true for $K_0(\mathbf{Var}_{/k})$ we may hope that it is true for $K(\mathbf{Var}_{/k})$ and that 5.1 is just the result of applying π_0. This in fact turns out to be the case.

Theorem 5.2. *The map of spaces*

$$K(\mathbf{Var}^{sm}_{/k}) \to K(\mathbf{Var}_{/k})$$

induced by the inclusion of categories $i : \mathbf{Var}^{sm}_{/k} \hookrightarrow \mathbf{Var}_{/k}$ *is a weak equivalence.*

Remark 5.1. This theorem is essentially Zakahrevich's [24], and may be stated in more generality, as hers is. However, we need a proof in this context. This will allow us to use merely *smooth* varieties to produce maps out of $K(\mathbf{Var}_{/k})$ — this will be quite useful because of the much nicer behavior of smooth varieties.

Remark 5.2. Note that the proof of additivity does not work for $\mathbf{Var}^{\mathrm{sm}}_{/k}$ because, for example, $\mathbf{A}^1_k \amalg_{\mathrm{Spec}\, k} \mathbf{A}^1_k$ is not smooth and so our conditions on pushouts do not hold in $\mathbf{Var}^{\mathrm{sm}}_{/k}$. Thus, the above also offers a proof of delooping of $K(\mathbf{Var}^{\mathrm{sm}}_{/k})$, which cannot be obtained via additivity, as it is for $K(\mathbf{Var}_{/k})$.

Proof. We use the $\widetilde{Q}$-construction and more or less mimic Quillen's proof of devissage in the case of abelian categories. The key here is that although $\mathbf{Var}_{/k}$ is nowhere near an abelian category, there is still all of the structure necessary (subobjects, etc).

We would like the induced map

$$N\widetilde{Q}\mathbf{Var}^{\mathrm{sm}}_{/k} \to N\widetilde{Q}\mathbf{Var}_{/k}$$

to be a weak equivalence. By Quillen's Theorem A [18], it is enough to show that $\widetilde{Q}i_{/X}$ for $X \in \mathbf{Var}_{/k}$ is contractible. To this end, we note that $\widetilde{Q}i_{/X} \simeq *$ if $X \in \mathbf{Var}_{/k}$ is smooth (since the category has a final object). If we can show that $\widetilde{Q}i_{/X} \to \widetilde{Q}i_{/X'}$ is a weak equivalence whenever $X \hookrightarrow X'$ is a closed inclusion and $X' - X$ is smooth we will be done. Indeed, we can consider a finite "resolution" until X is smooth, and the over-category is thus contractible.

So, let us consider $X \hookrightarrow X'$ where $X' - X$ is smooth. First, note that $\widetilde{Q}i_{/X'}$ is a category with objects $Y \xrightarrow{cl} Y' \xrightarrow{op} X'$. Let $J \subseteq \widetilde{Q}i_{/X'}$ be the subcategory where $Y \subset X$.

Then define two functors

(1) $s : \widetilde{Q}i_{/X'} \to J$ that takes

$$(Y \to Y' \to X') \mapsto (Y \cap X \to Y' \to X')$$

and this is left adjoint to the inclusion $i : J \hookrightarrow \widetilde{Q}i_{/X'}$. To see this, note that there are obvious unit and counit natural transformations $\mathrm{Id} \to s \circ i$ and $i \circ s \to \mathrm{Id}$.

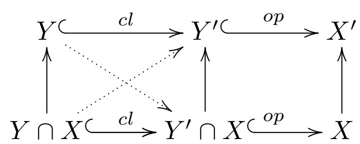
(2) $r : J \to \widetilde{Q}i_{/X}$ that takes

$$(Y \to Y' \to X) \mapsto (Y \to Y' \cap X' \to X')$$

and this is a right adjoint.

Since adjoints provide homotopies on geometric realization [18], we have $\widetilde{Q}i_{/X} \simeq J \simeq \widetilde{Q}i_{/X'}$. □

Remark 5.3. The argument is summed up in the diagram



6. Maps out of $K(\mathbf{Var}_{/k})$

We reap what we have sown above and produce maps out of $K(\mathbf{Var}_{/k})$. We will be able to lift various classical Euler characterstics including the point-counting map and Euler characteristics coming from cohomology theories. In addition, we produce a map to Waldhausen's K-theory of spaces.

In order to produce these maps, some work is needed. The spectrum $K(\mathbf{Var}_{/k})$ was of course constructed out of something that is *not quite* a Waldhausen category. As such, we need some language to deal with functors that "preserve" the various structures. This turns out to require some care. The issue is that maps on K-theory are usually produced by producing exact functors between the categories. Because of variance issues here, there will be issues with producing functors. We offer the following example, which illustrates the issues and will become relevant below.

Example 6.1. Consider the category of complex varieties $\mathbf{Var}_{/\mathbf{C}}$. There is a motivic measure $K_0(\mathbf{Var}_{/\mathbf{C}})$ to $\mathbf{Q}$-vector spaces obtained by taking compactly supported cohomology with $\mathbf{Q}$-coefficients. For "cofibration sequences" $Z \hookrightarrow X \hookleftarrow X - Z$ this procedure is covariant with respect to closed inclusions and contravariant with respect to open inclusions and yields long exact sequences

$$\cdots \to H^i_c(Z) \to H^i_c(X) \to H^i_c(X - Z) \to H^{i+1}_c(X) \to \cdots$$

and so if we assign

$$X \mapsto \chi(X) := \sum [H^i_c(X; \mathbf{Q})] \in K_0(\mathbf{Vect}_{\mathbf{Q}})$$

we get a well-defined motivic measure. But note: there was no functor that produced this. Taking compactly supported cohomology is contrvariant on open inclusions, but covariant on proper inclusions. Thus, to define a "derived" version of this Euler characteristic, another approach is needed.

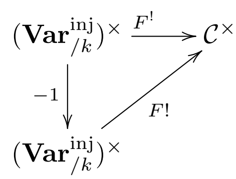
In general, to map $i\widetilde{S}_\bullet\mathbf{Var}_{/k}$ into other simplicial sets $wS_\bullet\mathcal{C}$ we will have to use functors that behave differently with respect to open and closed inclusions, because of the differences in vertical arrows in the respective $S_\bullet$-constructions. In fact, we'll have to deal with functors that are only *really* functors on the subcategory of closed inclusions and subcategory of open inclusions, respectively.

Remark 6.1. In what follows, it will be useful to consider the category $\mathbf{Var}^{\mathrm{inj}}_{/k}$ to be the category where the morphisms are monomorphisms.

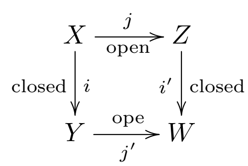
The definition below is inspired by proper base change theorems in algebraic geometry. It was suggested to the author by Jesse Wolfson. He also pointed out that it is quite close to [11, Defn. 3.3].

Definition 6.1. Let $\mathcal{C}$ be a Waldhausen category. We define a **pseudo-exact functor** from $\mathbf{Var}_{/k}^{\mathrm{inj}}$ to $\mathcal{C}$ to be a pair of functors $(F_!, F^!)$ such that

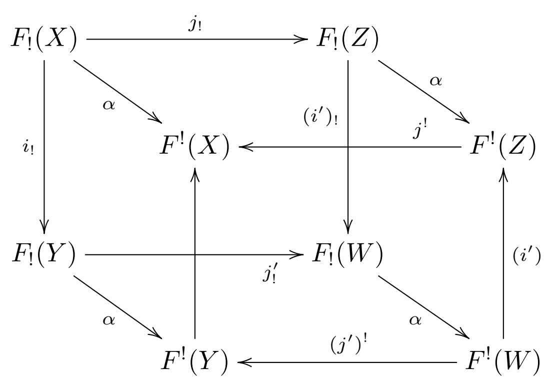
(1) $F_!$ is a functor $F_! : \mathbf{Var}_{/k}^{\mathrm{inj}} \to \mathcal{C}$
(2) $F^!$ is a functor $(\mathbf{Var}_{/k}^{\mathrm{inj}})^{\mathrm{op}} \to \mathcal{C}$
(3) Let $(-)^\times$ denote the internal groupoid and let (-1) denote inversion in that groupoid. Then there is a diagram with a natural transformation $\alpha : F^! \Longrightarrow F_! \circ (-)^{-1}$



(4) (**base change**)The diagram



produces a diagram



We require that

$$i_! \circ \alpha^{-1} \circ j^! \circ \alpha = \alpha^{-1} \circ (j')^! \circ \alpha \circ (i')_!$$

(5) (**cofiber**) For a cofiber sequence

$$X \overset{i}{\hookrightarrow} Y \overset{j}{\hookleftarrow} Y - X$$

the induced sequence

$$F_!(X) \xrightarrow{F_!(i)} F_!(Y) \xrightarrow{\alpha^{-1} F_!(j) \alpha} F_!(Y - X)$$

is a cofiber sequence in $\mathcal{C}$.

For ease, we will write a modified exact functor as $(F_!, F^!) : \mathbf{Var}_{/k} \to \mathcal{C}$ with the understanding that there is no underlying functor on the category $\mathbf{Var}_{/k}$.

We record the following consequence

Proposition 6.1. *Given a modified exact functor, there is a map of simplicial sets $i\widetilde{S}_\bullet \mathbf{Var}_{/k} \to wS_\bullet \mathcal{C}$ which induces a map of infinite loop spaces and thus spectra $K(\mathbf{Var}_{/k}) \to K(\mathcal{C})$.*

Proof. It is much easier to consider the proof by example. Consider a flag in $\widetilde{S}_3\mathcal{C}$ where we let $i_{a,b}$ denote the ath map at quotient level b, and $j_{a,b}$ denotes the map ath map from quotient level b to the one above (note that i is a closed inclusion and j is an open inclusion):

$$\begin{array}{ccccccc}
X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 \\
 & & \uparrow{\scriptstyle j_{1,0}} & & \uparrow{\scriptstyle j_{2,0}} & & \uparrow{\scriptstyle j_{3,0}} \\
 & & X_1 - X_0 & \xrightarrow{i_{1,0}} & X_2 - X_0 & \xrightarrow{i_{2,0}} & X_3 - X_0 \\
 & & & & \uparrow{\scriptstyle j_{2,1}} & & \uparrow{\scriptstyle j_{3,1}} \\
 & & & & X_2 - X_1 & \xrightarrow{i_{2,1}} & X_3 - X_1 \\
 & & & & & & \uparrow{\scriptstyle j_{3,2}} \\
 & & & & & & X_3 - X_2
\end{array}$$

We may apply $F_!$ to the closed inclusions and $\alpha^{-1}F^!\alpha$ to open inclusions to obtain the flags below.

$$\begin{array}{ccccccc}
F_!(X_0) & \xrightarrow{F_!(i_0)} & F_!(X_1) & \xrightarrow{F_!(i_1)} & F_!(X_2) & \xrightarrow{F_!(i_2)} & F_!(X_3) \\
 & & \downarrow{\scriptstyle \alpha^{-1}F^!(j_{1,0})\alpha} & & \downarrow{\scriptstyle \alpha^{-1}F^!(j_{2,0})\alpha} & & \downarrow{\scriptstyle \alpha^{-1}F^!(j_{3,0})\alpha} \\
 & & F_!(X_1 - X_0) & \xrightarrow{F_!(i_{1,0})} & F_!(X_2 - X_0) & \xrightarrow{F_!(i_{2,0})} & F_!(X_3 - X_0) \\
 & & & & \downarrow{\scriptstyle \alpha^{-1}F^!(j_{2,0})\alpha} & & \downarrow{\scriptstyle \alpha^{-1}F^!(j_{3,1})\alpha} \\
 & & & & F_!(X_2 - X_1) & \xrightarrow{F_!(i_{2,1})} & F_!(X_3 - X_1) \\
 & & & & & & \downarrow{\scriptstyle \alpha^{-1}F^!(j_{3,2})\alpha} \\
 & & & & & & F_!(X_3 - X_2)
\end{array}$$

This diagram commutes by part 4 of the axioms for a modified exact functor. Also, since $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$ and $F_!$ and $F^!$ are functors, it is clear that the simplicial maps are compatible. Furthermore, by the axioms we see that every map

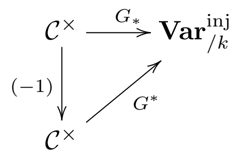
$$F_!(X_i) \to F_!(X_j) \to F_!(X_j - X_i)$$

is a cofiber sequence. Altogether, this means that the image flag is an object of $S_n\mathcal{C}$. □

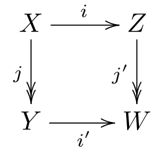
We also need a dual definition to prove maps *from* a Waldhausen category to $\mathbf{Var}_{/k}$. This situation seems to arise less commonly in practice, but will be useful in producing the splitting below 6.4.

Definition 6.2. Let $\mathcal{C}$ be a Waldhausen category. A **pseudo-op-exact functor** is a pair of functors (G_*, G^*) such that

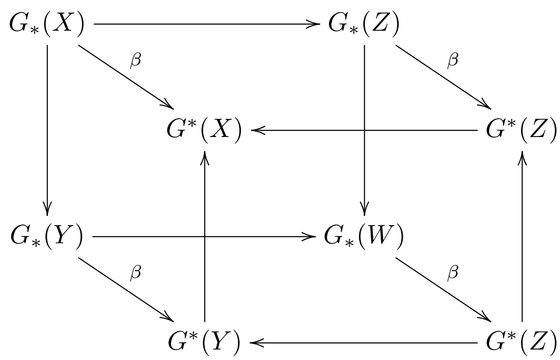
(1) G_* is a functor $G_* : \mathrm{co}(\mathcal{C}) \to \mathbf{Var}_{/k}^{\mathrm{inj}}$
(2) G^* is a functor $G^* : \mathrm{fib}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Var}_{/k}^{\mathrm{inj}}$
(3) There is a natural transformation $\beta : G_* \implies (-1) \circ G^*$ fitting in the following diagram



(4) Given a diagram



where the horizontal maps are cofibrations and vertical maps are fibrations, we get the corresponding diagram of k-varieties



We require that

$$i_* \circ \beta^{-1} \circ j^* \circ \beta = \beta^{-1} \circ (j')^* \circ \beta \circ (i')_*$$

(5) Given a cofiber sequence in $\mathcal{C}$

$$X \xrightarrow{i} Y \xrightarrow{j} Z$$

we get a cofiber sequence in $\mathbf{Var}_{/k}$

$$G_*(X) \xrightarrow{G_*(i)} G_*(Y) \xleftarrow{\beta G^* \beta^{-1}} G_*(Y - X)$$

Remark 6.2. Because of the rigidity of the category of varieties, these will be harder to produce in practice, in fact, the only example we know is the one below.

By a proof entirely dual to the one above, we obtain

Theorem 6.1. *Given a Waldhausen category $\mathcal{C}$ and a modified op-exact functor $\mathcal{C} \to \mathbf{Var}_{/k}$ we get a map on K-theory spectra*

$$K(\mathcal{C}) \to K(\mathbf{Var}_{/k}).$$

In the subsections below, we will have occasion to use the following category a number of times, so it worth defining before we get to work.

Definition 6.3. Let $\mathbf{FinSet}_+$ be the category of pointed finite sets. We choose a skeleton of it so that the objects are the pointed sets with n-elements $[\mathbf{n}]_+$. Morphisms are maps preserving the basepoint, which we denote $*$.

The salient property of this category for us is the following celebrated theorem.

Theorem 6.2 (Barratt-Priddy-Quillen)**.** *Consider* $\mathbf{FinSet}_+$ *as a Waldhausen category by defining cofibrations to be injective maps. Then*

$$K(\mathbf{FinSet}_+) \simeq S$$

Thus, $\mathbf{FinSet}_+$ will be our category-level model of the sphere spectrum.

Below it will be necessary to view $\mathbf{FinSet}_+$ as a Waldhausen category and also to understand some combinatorics.

First, we note that $\mathbf{FinSet}_+$ can be made into a Waldhausen category by declaring that cofibrations are monomorphisms and weak equivalences are isomorphisms. We record the following definition for future use.

Definition 6.4. A map $p : [\mathbf{n}_1]_+ \to [\mathbf{n}_2]_+$ will be said to be a **fibration** if it arises as a pushout

$$\begin{array}{ccc} [\mathbf{n}_0]_+ & \longrightarrow & [\mathbf{n}_1]_+ \\ \downarrow{\scriptstyle i} & & \downarrow \\ * & \longrightarrow & [\mathbf{n}_2]_+ \end{array}$$

where i is a cofibration. More concretely, p is a fibration if it is surjective and for $i \in [\mathbf{n}_2]_+$, $p^{-1}(i)$ has one element.

We define two flavors of wrong way maps in $\mathbf{FinSet}_+$.

Definition 6.5. Let $f : [\mathbf{n}_1]_+ \to [\mathbf{n}_2]_+$ be a monomorphism in $\mathbf{FinSet}_+$. We define $f^* : [\mathbf{n}_2]_+ \to [\mathbf{n}_1]_+$ by mapping the corange to the basepoint and each $i \in \mathrm{Im}(f)$ to $f^{-1}(i)$.

Definition 6.6. Let $p : [\mathbf{n}_1]_+ \to [\mathbf{n}_2]_+$ be a fibration. We define p^* as follows. For $i \in \mathrm{Im}(p)$, define $p^*(i) = p^{-1}(i)$ and then map the basepoint to the basepoint.

We now consider commutative diagrams

$$\begin{array}{ccc} [\mathbf{n}_1]_+ & \hookrightarrow & [\mathbf{n}_2]_+ \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow \\ [\mathbf{n}_3]_+ & \hookrightarrow & [\mathbf{n}_4]_+ \end{array}$$

What does it mean for these diagrams to be commutative? It will mean that $p_1^{-1}(*) = i_1^{-1}(p_2^{-1}(*))$ and that for $i \in [\mathbf{n}_4]_+$, $(i_2 \circ p_1)^{-1}(i) = (p_2 \circ i_1)^{-1}(i)$.

This observation has the following simple, but useful, consequence.

Lemma 6.1. *Given a commutative diagram as above, the following also commutes*

$$\begin{array}{ccc} [\mathbf{n}_1]_+ & \xrightarrow{i_1} & [\mathbf{n}_2] \\ \uparrow p_1^* & & \uparrow p_2^* \\ [\mathbf{n}_3]_+ & \xrightarrow[i_2]{} & [\mathbf{n}_4]_+ \end{array}$$

6.1. **The Unit Map.** Since it is a spectrum, $K(\mathbf{Var}_{/k})$ naturally has a unit map from the sphere spectrum $S \to K(\mathbf{Var}_{/k})$. It will be useful for us to have a model for this. When working with K-theoretic functors, finite pointed sets are always a proxy for the sphere spectrum, by Barrat-Priddy-Quillen. We construct functors out of this cateogry to model maps out of the sphere spectrum.

Definition 6.7. We define a modified op-exact functor $(G_*, G^*) : \mathbf{FinSet}_+ \to \mathbf{Var}_{/k}$ as follows.

(1) $G_* : \mathbf{FinSet}_+ \to \mathbf{Var}_{/k}^{\text{inj}}$ is defined on objects by

$$G_*([\mathbf{n}]_1) = \coprod_{i=0}^{n_1} \operatorname{Spec}(k).$$

One cofibrations, i.e. inclusions it is defined by the corresponding inclusions of of Spec(k)s. On fibrations, it is defined by the corresponding fold maps.

(2) $G^* : \mathbf{FinSet}_+ \to \mathbf{Var}_{/k}^{\text{inj}}$ is defined by objects as above. Given a cofibration $i : [\mathbf{n}_1]_+ \to [\mathbf{n}_2]_+$, we define G^* to be $G_*(i^*)$ with i^* defined as in 6.5. Given a fibration $p : [\mathbf{n}_1]_+ \to [\mathbf{n}_2]_+$ we define $G^*(p)$ to be $G_*(p^*)$ with p^* defined as in 6.6.

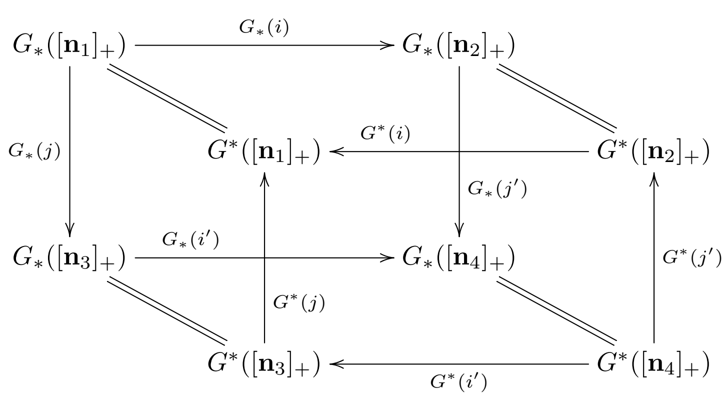
We check that this is an honest modified op-exact map.

Proposition 6.2. *The map above is in fact modified exact.*

Proof. The first 3 axioms are trivial. To check the 4th, we consider a diagram in $\mathbf{FinSet}_+$

$$\begin{array}{ccc} [\mathbf{n}_1]_+ & \xrightarrow{i} & [\mathbf{n}_2]_+ \\ j \downarrow & & \downarrow j' \\ [\mathbf{n}_3]_+ & \xrightarrow[i']{} & [\mathbf{n}_4]_+ \end{array}$$

This induces a diagram of varieties



We now check that the two maps we need to agree in fact agree. That is, we need

$$G^*(i') \circ G_*(j') = G_*(j) \circ G^*(i).$$

However, this is the content of Lem 6.2.

□

Thus, Prop. 6.1 gives

Corollary 6.1. *There is a map* $S \to K(\mathbf{Var}_{/k})$*.*

Remark 6.3. It is not hard to see that $\mathbf{FinSet}_+ \to \mathbf{Var}_{/k}$ is in fact pseudo-multiexact, yielding an E_∞ map $S \to \mathbf{Var}_{/k}$.

6.2. **Point Counting.** The first Euler characteristic we lift is point-counting. This is one of the most basic and interesting Euler characteristics. For the rest of this subsection, we assume that k is a finite field.

Definition 6.8. Define a modified exact functor $(\mathbf{pc}_!, \mathbf{pc}^!) : \mathbf{Var}^{\mathrm{inj}}_{/k} \to \mathbf{FinSet}_+$ as follows.

On objects, we define $\mathbf{pc}(X)$ to be the *set* of k-points of X and then assigning an order to them, once and for all.

For a map $f : X \to Y$, define $\mathbf{pc}_!(f)$ via postcomposition. That is, given the set of k-points $\{\mathrm{Spec}(k) \to X\}$, map them to their corresponding $\{\mathrm{Spec}(k) \to Y\}$.

For $f : X \to Y$ define $\mathbf{pc}^!(f)$ by restriction. That is, given a k-point $p : \mathrm{Spec}(k) \to Y$, if p has a lift as follows

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & Y \end{array}$$

we define $\mathbf{pc}^!(p)$ to be the k-point of X in the lift. If it does, we define it to be the basepoint.

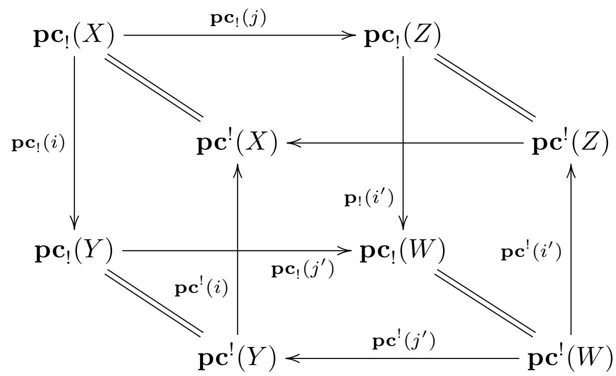
The natural tranformation α is taken to be the identity.

Proposition 6.3. *The functors* $\mathbf{pc}^!$ *and* $\mathbf{pc}_!$ *assemble to a pseudo-exact functor* $(\mathbf{pc}_!, \mathbf{pc}^!) : \mathbf{Var}_{/k} \to \mathbf{FinSet}_+$*.*

Proof. We need to verify the conditions of Def. 6.1. Suppose we have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ Y & \xrightarrow[j']{} & W \end{array}$$

where j, j' are closed and i,i' are open. Then we get an induced cube in $\mathbf{FinSet}_+$



and we would like to check that

$$\alpha^{-1} \circ \mathbf{pc}^!(j') \circ \alpha \mathbf{pc}_!(i') = \mathbf{pc}_!(i) \circ \alpha^{-1} \circ \mathbf{pc}^!(j) \circ \alpha.$$

However, this is a consequence of Lem. . □

Note that if we have two k-varieties X, Y then the number of k-points in $X \times_k Y$ is the product of the number of k-points in X and Y. This product can be made functorial.

Theorem 6.3. *There is a map of E_∞ ring spectra $K(\mathbf{Var}_{/k}) \to S$*

Proof. Apply K theory to the above. By Barrat-Priddy-Quillen [20], $K(\mathbf{FinSet}_+) \simeq S$. □

Proposition 6.4. *The composition of the point-cointing map with the unit map is the idenity, thus the sphere spectrum splits off of $K(\mathbf{Var}_{/k})$ and we may write $K(\mathbf{Var}_{/k}) \simeq S \vee \widetilde{K}(\mathbf{Var}_{/k})$.*

Proof. We consider the compostion of pseudoexact and pseudo-op-exact functors

$$\mathbf{FinSet}_+ \xrightarrow{(G_*, G^*)} \mathbf{Var}_{/k} \xrightarrow{(\mathbf{pc}_!, \mathbf{pc}^!)} \mathbf{FinSet}_+.$$

It is easy to see that this is the identity. Upon taking the corresponding $S_\bullet$ and $\widetilde{S}_\bullet$ constructions, this gives a splitting

$$S \to K(\mathbf{Var}_{/\mathbf{C}}) \to S.$$

□

6.3. **Map to Waldhausen A-Theory.** Throughout this subsection we work over the base field $\mathbf{C}$. In this case varieties may be considered as topological spaces. However, there is already a K-theory of topological spaces, namely, Waldhausen's $A(*)$ [22]. We produce a map $K(\mathbf{Var}_{/\mathbf{C}}) \to A(*)$ relating these two K-theories.

First, we recall the definition of Waldhausen's $A(*)$. We opt for the more geometric definition. We could have also treated $A(*)$ as the algebraic K-theory of the ring spectrum S.

Definition 6.9. We let $\mathcal{R}^f_{/*}$ denote the category of topological spaces, X, provided with finite CW-structure and maps $r : X \to *$ and $s : * \to X$ such that $sr = \mathrm{Id}$.

The category $\mathcal{R}^{f}_{/*}$ is endowed with a Waldhausen structure given by the usual cofibrations and weak equivalences homotopy equivalences. Then

$$A(*) = \Omega |wS_{\bullet}\mathcal{R}^{f}_{/*}|.$$

Waldhause also produces another model of $A(*)$, which will be more useful for us.

Definition 6.10. Let $\mathcal{R}^{hf}_{/*}$ be the Waldhausen category of homotopy finite retractive spaces. These are spaces homotopy equivalent to a finite complex, equipped with cofibrations given by the homotopy extension property and weak equivalences the usual weak equivalences.

We need to produce a pseudoexact map $\mathbf{Var}^{sm}_{/\mathbf{C}} \to \mathcal{R}^{hf}_{/*}$. First, there is a forgetful functor $\mathbf{Var}^{sm}_{/C} \to \mathbf{Top}$ given by considering the smooth variety as a topological space. We show below that such a topological space, once we take one-point compactification, is homotopy finite.

Proposition 6.5. *Consider X a smooth, complex variety. If we consider it as a topological space and consider the one point compactification X^+, then X^+ is a homotopy equivalent to a finite CW-complex.*

Proof. A smooth complex variety is a smooth complex manifold, which by forgetting is a smooth real manifold. As an application of Morse theory, this can be decomposed into cells, and some possibly unstable cells. Now, upon taking one point compactification, the unstable cells will all become cells. Thus, X is a CW-complex. Since this arises from an algebraic function, this smooth manifold has only a finite number of critical points and is thus a finite CW-complex. □

Proposition 6.6. *The one-point compactification functor $(-)^+$ is a pseudoexact functor $((-)^+_!, (-)^{+,!}) : \mathbf{Var}^{sm}_{/\mathbf{C}} \to \mathcal{R}^{hf}_{/*}$ is pseudo-exact.*

Proof. One point compactification is covariant with respect to proper maps between topological spaces and covariant with respect to open inclusions. The necessary diagrams obviously commute. □

Combining the above proposition with Thm. 5.2 we obtain

Theorem 6.4. *There is a map of spectra*

$$K(\mathbf{Var}_{/\mathbf{C}}) \to A(*)$$

Remark 6.4. The homotopy groups and homotopy type of $A(*)$ have recently been given expression in [7]. It would be very interesting to know what parts of this are picked up by $K(\mathbf{Var}_{/C})$.

Remark 6.5. Using trace methods there is a map $A(*) \to S$, and thus a composition

$$K(\mathbf{Var}_{/\mathbf{C}}) \to A(*) \to S.$$

This is likely the analgoue of point-counting or the Euler characteristic.

6.4. **Lifted Euler Characteristic.** We proceed to lift Euler characteristics that arise from cohomology theories. So that we do not have to deal with technicalities involving ℓ-adic sheaves, we restrict our attention in this section to the ground field $k = \mathbf{C}$. In this case, the varieties are topological spaces, and if they are smooth then they are in fact manifolds. We then define $H_c(X; \mathbf{Q})$ (or $H_c(X; \mathbf{R})$ or $H_c(X; \mathbf{C})$) in the usual way.

Definition 6.11. Let $C_c^*(X; \mathbf{Q})$ denote a chain complex that computes the compactly supported cohomology $H_c^*(X; \mathbf{Q})$ — for concreteness singular cochains with compact support.

One defines a functor $X \to \mathbf{Ch}^{hb}(\mathbf{Q})$ by assigning X to $C_c^*(X; \mathbf{Q})$. Given an open immersion $X \hookrightarrow Y$ one gets a map $C_c^*(Y; \mathbf{Q}) \to C_c^*(X; \mathbf{Q})$ and given a proper map $X \to Y$ we get a map $C_c^*(X; \mathbf{Q}) \to C_c^*(Y; \mathbf{Q})$. In particular, closed inclusions $X \hookrightarrow Y$ are proper.

The following is a restatement of a standard fact about compactly supported cohomology.

Proposition 6.7. *The above is a pseudoexact functor from $\mathbf{Var}_{/\mathbf{C}}$ to homologically bounded chain complexes, denoted $\mathrm{Ch}^{hb}_{\mathbf{Q}}$.*

We now produce our map of spectra.

Theorem 6.5. *There is a map of spectra*

$$K(\mathbf{Var}_{/C}) \to K(\mathbf{Q})$$

and on π_0 this corresponds to the usual Euler characteristic, $K_0(\mathbf{Var}_{/\mathbf{C}}) \to \mathbf{Z}$.

Proof. The proposition above produces a map

$$K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathrm{Ch}^{hb}_{\mathbf{Q}})$$

by, for example, the $\widetilde{Q}$-construction and the Q-construction. Now, by Gillet-Waldhausen (see, for example, [23]), there is a weak equivalence $K(\mathbf{Q}) \simeq K(\mathrm{Ch}^{b}_{\mathbf{Q}})$ which is given by the Euler characteristic of complexes. Thus, we have a composite

$$K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathrm{Ch}^{hb}_{\mathbf{Q}}) \xrightarrow{\chi, \simeq} K(\mathbf{Q}).$$

That it corresponds to the usual Euler characteristic is a consequence of the latter isomorphism and the fact that the Euler characteristic of a chain complex is equal to the Euler characteristic of it's corresonding homology. □

Remark 6.6. Of course, we chose the field to work over rather arbitrarily. We could have chosen $\mathbf{R}$ or $\mathbf{C}$ and obtained maps

$$K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathbf{C})$$
$$K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathbf{R})$$

Via considering $\mathbf{C}$ with the discrete topology $\mathbf{C}^\delta$, this also gives a map

$$K(\mathbf{Var}_{/C}) \to ku.$$

We also have the following proposition, which is a consequence of how compactly supported cohomology is defined.

Proposition 6.8. *The map $K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathbf{Q})$ factors through $A(*)$:*

$$K(\mathbf{Var}_{/\mathbf{C}}) \to A(*) \to K(\mathbf{Q}).$$

Remark 6.7. It is possible that one could incorporate Hodge structures into the above discussion to get a more refined map. In the case of the Grothendieck ring $K_0(\mathbf{Var}_{/k})$ one can certainly produce a map $K_0(\mathbf{Var}_{/k}) \to K_0(\mathbf{HS})$ where $\mathbf{HS}$ is the (abelian) category of Hodge structures by simply applying the functor $H_c^*(-;\mathbf{Q})$. However, since this requires passing to cohomology, it is not clear how to refine this to a chain level construction. We leave this for future work.

7. Conjectures and Future Work

This paper has set up a model for investigating $K(\mathbf{Var}_{/k})$. There are of course further points to investigate. Not only are there most likely many more derived motivic measures, but one may wonder about the relationship with other aspects of $K_0(\mathbf{Var}_{/k})$, for example, whether motivic integration could be lifted.

Let us briefly discuss a conjectural motivic measure. When looking for a motivic measure, we of course have to produce pseudo-exact fuctors, and thus need functors with certain specific variance properties. We consider one such functor presently.

Let X be a Noetherian scheme. Quillen defines $K'(X)$ to be $K(\mathbf{Coh}(X))$, that is he defines it to be the K-theory of the abelian category of coherent sheaves on X [18]. He also proves the following proposition

Proposition 7.1. [18, 3.1] *Let $X \hookrightarrow Y$ be a closed immersion. Then there is a cofibration sequence of spectra*

$$K'(X) \to K'(Y) \to K'(Y - X)$$

This means that $K'(-)$ is exactly the sort of functor that we like. It is covariant with respect to closed inclusions, and contravariant with respect to open inclusions. It thus gives us a pseudo-exact functor $K' : \mathbf{Var}_{/k} \to \mathbf{Sp}$ where the latter is the category of spectra considered as a Waldhausen category via its model structure. Now, for regular schemes, $K'(X) \simeq K(X)$. Since we may restrict our attention to smooth varieties, we may assume regularity. Furthermore, every K-theory spectrum $K(X)$ is a $K(S)$-module. Thus the K' functor is actually a(n) (exact) functor

$$K' : \mathbf{Var}^{\mathrm{sm}}_{/k} \to \mathbf{Mod}_{K(S)}$$

where the latter denotes the modules over the E_∞-ring $K(S)$. We would like this to produce a map on K-theory. However, by the Eilenberg swindle, the K-theory of $\mathbf{Mod}_{K(S)}$ vanishes. In order to get a map $K(\mathbf{Var}_{/k}) \to K(K(S))$ we require that K' land in *compact* $K(S)$-modules. To put this more succinctly, we have two conjectures, the former implied by the latter.

Conjecture 3. *There is a map of ring spectra*

$$K(\mathbf{Var}_{/k}) \to K(K(S))$$

Conjecture 4. *Let X be a smooth scheme. Then $K(X)$ is compact as a $K(S)$-module.*

Even if the latter doesn't obtain, it seems likely that the former will — it's just the most direct route to the result.

Remark 7.1. When X is a k-variety, $K'(X)$ is also a $K(k)$-module. It is also possible that $K'(X)$ could be compact as a $K(k)$-module, in which case we would have a map

$$K(\mathbf{Var}_{/k}) \to K(K(k))$$

Remark 7.2. The ring $K(K(S))$ — doubly iterated K-theory — is conjectured to have a deep relationship with chromatic homotopy theory. We leave any speculation about what part of this $K(\mathbf{Var}_{/k})$ might see to others.

We come back into a more reasonable orbit and offer some more tractable conjectures. Various derived motivic measures offer ways of probing classes in $K(\mathbf{Var}_{/k})$. For example, the following conjecture seems reasonable

Conjecture 5. *The map* $K(\mathbf{Var}_{/\mathbf{C}}) \to K(\mathbf{Q})$ *is surjective.*

It certainly is on π_0. Furthermore, we know a fair amount about the homotopy groups of $K(\mathbf{Q})$. In particular, we know $\pi_1 K(\mathbf{Q}) = \mathbf{Z}/2$. It thus seems quite reasonable to ask exactly what the map

$$K_1(\mathbf{Var}_{/k}) \to \pi_1 K(\mathbf{Q}) = \mathbf{Z}/2$$

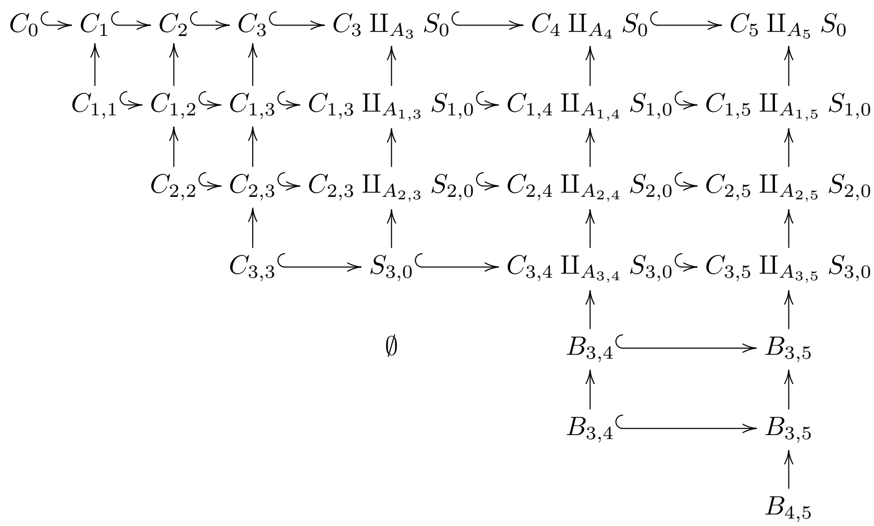
is measuring. We hope to address this in future work.

8. Appendix: Simplicial Homotopy

In this appendix we present a few diagrams to aid in intuition with the simplicial homotopy produced in the additivity theorem. The simplex $h_3(e)$ where e is a 5-simplex looks like

$$\begin{array}{ccccccccccccc}
A_0 & \hookrightarrow & A_1 & \hookrightarrow & A_2 & \hookrightarrow & A_3 & \hookrightarrow & S_0 & = & S_0 & = & S_0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & A_{1,1} & \hookrightarrow & A_{1,2} & \hookrightarrow & A_{1,3} & \hookrightarrow & S_0 - A_1 & = & S_0 - A_1 & = & S_0 - A_1 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & & A_{2,2} & \hookrightarrow & A_{2,3} & \hookrightarrow & S_0 - A_2 & = & S_0 - A_2 & = & S_0 - A_2 \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & & & & A_{3,3} & \hookrightarrow & S_0 - A_3 & = & S_0 - A_3 & = & S_0 - A_3 \\
 & & & & & & & & & & & & \\
 & & & & & & & & \emptyset & & \emptyset & & \emptyset \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & \emptyset & & \emptyset \\
 & & & & & & & & & & & & \\
 & & & & & & & & & & & & \emptyset
\end{array}$$

The more important part of the simplicial homotopy is the $h_i^C(e)$ simplex. The ppicture below is of $h_3(e)$ when e is a 5-simplex. For compactness we write $S_{i,0} := S_0 - A_i$.



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