**OBSTRUCTIONS TO RECTIFYING HOMOTOPY COMMUTATIVE DIAGRAMS**

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Contents



## 1. Warm-Up : The Case of Groups

Here's the problem we're face with. We have a "homotopy action" of a group $G$ on a space, $X$, and we want to know when it can be rectified into an actual topological action of $G$ on the space $X$. That is, when it can be made to act through homeomorphisms. Given the prescence of the quotes in the above, it should be clear that some definitions are need. Before we can define what we mean by homotopy action and rectified we need to define a few spaces that will be useful to us.

**Remark.** I should say now that this is a shadow of a much, much richer phenomenon. Namely, the phenomen on that "homotopy coherence" is just as good as "on the nose". In most cases, homotopy coherence can be rectified to "on the nose". The result Cooke proves is a baby example of that.

In what follows all spaces will be CW complex (i.e. cofibrant)

**Definition.** The space $\mathrm{haut}(X)$ is the space of homotopy automorphisms $X \to X$ topologized as a subspace of $X^X$. It is an $H$-space.

**Definition.** The group $[\mathrm{haut}(X)]$ will be $\pi_0 \, \mathrm{haut}(X)$. It's the group of homotopy automorphisms up to homotopy equivalence.

We can now define a homotopy action:

**Definition.** A **homotopy action** of group $G$ on a space $X$ is a map $G \to [\mathrm{haut}(X)]$.

Of course, such a thing, we want to know about a notion of "equivalence".

**Definition.** Two homotopy actions are **equivalent** if there is a homotopy equivalence $f : X \to Y$ and a commutative diagram

$$\begin{array}{ccc} & & [\mathrm{haut}(X)] \\ & \nearrow & \downarrow \\ G & & \\ & \searrow & \\ & & [\mathrm{haut}(Y)] \end{array}$$

The map downward is given by $[g] \mapsto [fgf^{-1}]$.

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**Definition.** A homotopy action can be **rectified** if there is a homotopy equivalence $f : X \to Y$, and a diagram as a bove, where $G \to [\mathrm{haut}(Y)]$ factors through homeomoprhisms.

In the course of the proof of the main theorem we're going to need some facts about classifying spaces. Recall that Stasheff constructed classifying spaces for any $H$-space. Recall also

**Definition.** An $X$-fibration over a (pointed) space $B$ is a fibration $E \to B$ together with a homotopy equivalence $f_E : X \to p^{-1}(*)$. So an $X$-fibration is a pair $(p, f_E)$.

**Proposition.** *$X$-fibrations are classfied by maps $B \to B\,\mathrm{haut}(X)$.*

We're now ready to state the theorem :

**Theorem** (Cooke)**.** *Let $\alpha : G \to [\mathrm{haut}(X)]$ be a homotopy action. This can be rectified to a topological action if there is a solution to the lifting problem:*

$$\begin{array}{ccc} & & B\,\mathrm{haut}(X) \\ & \nearrow & \downarrow \\ BG & \xrightarrow[B\alpha]{} & B[\mathrm{haut}(X)] \end{array}$$

**Remark.** What this says is that once we get to the point where we have something richer than our anemic homotopy action, we get an actual action.

**Corollary.** *We can apply standard obstruction theory. We get that the obstructions to lifting lie in*

$$H^n(G, \{\pi_{n-2}(\mathrm{haut}(X)_{id})) \qquad n \geq 3$$

**Corollary.** *Actions can always be rectified for free groups.*

**Remark.** Of course, in practice, the above are horrible to compute. Cooke gives a large class of spaces for which the above vanishes for abstract reasons. We'll do this below.

Now time for the proof, which is not all that hard.

*Proof.* First, let's assume that we actually have rectified the action. We'll produce a lift. To this end, suppose we have a homotopy equivalence $f : X \to Y$ and an action of $G$ on $Y$, compatible with the (homotopy) action of $G$ on $X$ in the sense above.

We can assume the action of $G$ on $Y$ is free. We can do this since we're allowed to just cross any space with a contractible free $G$-space. So now it makes sense to talk about $Y \to Y/G$. This is a fibration with fiber $G$,so it's classified by $Y/G \xrightarrow{\theta} BG$. This map itself can be made into a fibration $\overline{\theta} : E \to BG$. Of course now, there is a homotopy equivalence $\overline{\theta}^{-1}(*) \simeq Y$ and thus there is a homotopy equivalence $\widetilde{f} : X \to \overline{\theta}^{-1}(*)$. The homotopy action on $\overline{\theta}^{-1}(*)$ is compatible in the sense we've specified to the action of $G$ on $X$.

Of course, now we have an $X$-fibration and so it's classified by a map $BG \to B\,\mathrm{haut}(X)$. We've thus produced a lifting

$$\begin{array}{ccc} & & B\,\mathrm{haut}(X) \\ & \nearrow & \downarrow \\ BG & \longrightarrow & B[\mathrm{haut}(X)] \end{array}$$

Now we go the other way. Suppose we produced such a lifting, that is, we've been given a map

$$\varphi : BG \to B\,\mathrm{haut}(X)$$

We want to show that we can get a slightly richer structure out of this - i.e. an actual action. To do so, we consider the pullback diagram

$$\begin{array}{ccc} \varphi^*(E\,\mathrm{haut}(X)) & \longrightarrow & E\,\mathrm{haut}(X) \\ \downarrow & & \downarrow \\ BG & \longrightarrow & B\,\mathrm{haut}(X) \end{array}$$

The thing on the left is a fibration over $BG$ with fiber the homotopy type of $X$. Now we tilt it on it's side and sconsider the pullback

$$\begin{array}{ccc} Y & \longrightarrow & EG \\ \downarrow & & \downarrow \\ \varphi^*(E\,\mathrm{haut}(X)) & \longrightarrow & BG \end{array}$$

$Y$ has the homotpy type of $X$ (it fits into a fiber sequence that $X$ also fits into), $G$ obvious acts on $Y$. □

**Remark.** For my own purposes the arguments can be summarized succinctly:

$$\begin{array}{ccc} Y \to Y/G & & BG \to B\,\mathrm{haut}(X) \\ \wr & & \wr \\ Y/G \to BG & & \varphi^*(E\,\mathrm{haut}(X)) \to BG \\ \wr & & \wr \\ E \to BG & & Y \to \varphi^*(E\,\mathrm{haut}(X)) \\ \wr & & \\ BG \to \mathrm{haut}(X) & & \end{array}$$

1.1. **Some Cases where the Obstructions Vanish.** We know some cases where group cohomolgy vanishes:

**Proposition.** *$G$ a group of order $n$. Suppose $M$ is a $G$-module. Then, $H^i(G, M)$ is an $\mathbf{Z}/n$-module. (for $i \neq 0$)*

Note that $i \neq 0$ doesn't matter, since our obstructions live in higher degrees anyway.

**Corollary.** *If $\cdot n : M \to M$ is an isomorphism, then $H^i(G, M) = 0$.*

So now we want to look for cases where the coefficient modules $\{\pi_{i-2}\,\mathrm{haut}(X)_{\mathrm{Id}}\}$ vanish for abstract reasons.

To do this we'll deal with $h_*$-local spaces for some homology theory $h_*$. It's often a bit easier to control the homotopy groups of these spaces, and the homotopy groups of maps out of these spaces.

We make the following definition:

**Definition.** For a homology theory $h_*$, let $\mathbf{P}_{h_*}$ be the set of primes such that multiplication by $p$ is an isomoprhism on $h_*(pt)$.

**Proposition.** *$X$ connected and $h_*$-local. Assume $q \in \mathbf{P}_{h_*}$. Then $\pi_i(X)$ is uniquely $q$-divisible.*

*Proof.* Consider $S^i \to S^i$ the degree $q$ map. It induces $h_*(S^i) \to h_*(S^i)$ which is multiplication by $q$ on and therefore an isomorphism, thus $S^i \to S^i$ is an $h_*$-equivalence. By definition of being $h_*$-local, we thus have

$$[S^i, X] \to [S^i, X]$$

is a bijection. Since $S^i \to S^i$ is the degree $q$ map, the map above is multiplication by $q$, and it's a bijection,s o $\pi_i(X)$ is uniquely $q$-divisible. □

We can say a little about mapping spaces too:

**Proposition** (Bousfield)**.** *$X, Y$ connected. If $Y$ is $h_*$-local, then so is each component of* $\mathrm{Hom}(X, Y)$.

**Corollary.** *Let $v : X \to Y$ be a map. If $Y$ is $h_*$-local and $q \in \mathbf{P}_{h_*}$, then* $\mathrm{Hom}(X, Y)$ *is $h_*$-local, and thus*

$$\pi_i(\mathrm{Hom}(X, Y), v)$$

*is uniquely $q$-divisible.*

**Corollary.** *In particular for $X$ $h_*$-local and* $\mathrm{Id} : X \to X$, $\pi_i(\mathrm{Hom}(X,X), \mathrm{Id})$ *is uniquely $q$-divisble.*

These all lead up to:

**Theorem** (Cooke)**.** *$G$ a group of order $n$ and suppose that multiplication by $n$ is an iso of $h_*(pt)$. Suppose also that $X$ is $h_*$-local. Then any homotopy action can be rectified.*

*Proof.* The obstructions to rectification live in $H^i(G; \{\pi_{i-2}(\mathrm{haut}(X)_{\mathrm{Id}}, 1)), i \geq 3$. But $\mathrm{haut}(X)_{\mathrm{Id}}$ is the identity component of $\mathrm{Hom}(X, X)$. And that guys homotopy groups are uniquely $q$-divisible for all $q$ dividing $n$. □

1.2. **Non-Vacuity.** Cooke provides an example of a homotpy action that cannot be rectified. I don't really want to go through the example.

1.3. **Some Cute Applications.** Before we go on to massive generality, it's worth writing down some consequences :

Here's a question : We have a homotopy $G$ action. It induces a $G$-action on $h^*(X)$. So, we can reasonable speak of $h^*(X)^G$. Is there a space that realizes this cohomology group? That is, is there $X_\alpha$ such that $h^*(X_\alpha) = h^*(X)^G$.

We can answer that kind of quickly:

Suppose $h_* = H_*(-; R)$ with $R = \mathbf{Z}/p$ or $R \subset \mathbf{Q}$.

**Theorem** (Cooke)**.** *Let $\alpha : G \to [\mathrm{haut}(X)]$ be a homotopy action. Suppose $|G|$ is invertible in $R$. Then there exists $X_\alpha$ and a map $f : X \to X_\alpha$ such that $f^* : h^*(X_\alpha) \to h^*(X)$ has image $h^*(X)^G$.*

*Proof.* $\alpha : G \to [\mathrm{haut}(X)]$ induces $E\alpha : G \to [\mathrm{haut}(EX)]$ (since the localization functor is functorial). Now, $EX$ is $h_*$-local, and $G$ satisfies the conditions of the theorem above, so we can rectify to an actual action $G$ on a space $Y \simeq EX$. Assume the action of $G$ on $Y$ is free. We then get $Y \to Y/G := X_\alpha$ a covering space.

We get $h^*(X_\alpha) \to h^*(Y)$ mapping onto $h^*(Y)^G$. Then $X \to EX \to Y$ is an $h_*$-equiv and we're done. □

Now, once assumes that $[\mathrm{haut}(X)]$ is actually a rather complicated object, since you'd expect $\mathrm{haut}(X)$ to be. Note that there is a map (of groups)

$$[\mathrm{haut}(X)] \to \mathrm{Aut}\, h_*(X).$$

This is sort of telling us which homotopy automorphisms are "seen" with homology. We can ask about the kernel of this map:

$$\mathcal{E}_1(X) \to [\mathrm{haut}(X)] \to \mathrm{Aut}\, h_*(X)$$

Amazingly, we can say something about this in local cases!

Let $h_*$ be as above.

**Theorem.** *Let $\mathbf{P}$ denote the set of primes which are not invertible in $R$ and suppose $X$ is $h_*$-local. Then $\mathcal{E}_1(X)$ only has $\mathbf{P}$-torsion.*

*Proof.* Let $f \in \mathrm{haut}(X)$ such that

(1) $f^q \simeq \mathrm{Id}$ where $q \notin \mathbf{P}$
(2) $f_* = 1$.

$f$ specifies a homotopy $G$-action. It can be rectified. So, by the usual tricks we get a free topological action of $G$ on $X$ and we can consider

$$X \to X/G$$

By assumption, this is the identity on $h_*$ (assumption 2). Thus, $X \to X/G$ is an $h_*$-equivalent, and so applying the localization functor we have a homotopy equivalence $EX \to E(X/G)$. We can consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & EX \\ \downarrow & & \downarrow \simeq \\ X/G & \longrightarrow & E(X/G) \end{array}$$

Thus, $X \to E(X/G)$ is a homotopy equivalence. So,

$$X/G \to E(X/G) \simeq X$$

provides a homotopy inverse to $X \to X/G$. So, $X \simeq X/G$ and thus $f \simeq 1$.

□

## 2. Generalization

Before I get into the specifics of how this generalizes to diagrams, I want to illustrate some of the problems we'll have to face in generalizing.

(1) What is a homotopy commutative diagram? This much is a clear. Fine. What is the the equivalent of $G \to \mathrm{haut}(X)$ in the case of diagrams? The answer turns out to be what Dwyer-Kan call an $\infty$-commutative diagram - you build all higher homotopy into the diagram
(2) What is the proper notion of rectificatino in this case? Dwyer-Kan call it **realization**, and there is a whole category of realizations. One has to be a bit careful about the equivalences that must be introduced.
(3) Dwyer-Kan prove that once you have this $\infty$-commutative diagram, i.e. one which is a bit richer than our initial homotopy diagram, we get a realization. Even better they show that the space of $\infty$-commutative diagrams is equivalent to the space of realizing diagram. This is not easy.
(4) What is the appropriate notion of obstruction? We know about obstruction theory for spaces - but not so much for diagrams or categories. Dwyer-Kan build cohomology theories for simplicial sets and simplicial categories. The obstruction to lifting an $n$-commutative diagram to an $(n+1)$-commutative diagram lives in these cohomology theories.

**Remark.** Lurie proves an incredibly strong result about rectification in Higher Topos Theory. He shows that given any small category $\mathcal{I}$ and any $\infty$-category $\mathcal{C}$ and a digram $c : N(\mathcal{I}) \to \mathcal{C}$ we can choose a fibrant simplicial category $\mathbf{C}$ and an equivalence $e : N(\mathbf{C}) \leftarrow \mathcal{C}$ and a map $d : \mathcal{I} \to \mathbf{C}$ such that $N(d)$ and $e \circ c$ are equivalent as objects of $\mathrm{Fun}(N(\mathcal{I}), N(\mathbf{C}))$.

This is more or less the statement that once we have homotopy coherence, we can rectify.

I think the Dwyer-Kan result is slightly more general - it says something about the homotopy type of realizations and $\infty$-commutative diagrams.